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On some geometric and dynamical aspects of discrete and locally compact groups

présentée par

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Contents

Works presented

Chapter 1:

- Piecewise strongly proximal actions, free boundaries and the Neretin groups (with P.-E. Caprace and N. Matte Bon).
  Bull. Soc. Math. France, Tome 150, Fascicule 4, 2023, pp. 773-795.
- A commutator lemma for confined subgroups and applications to groups acting on rooted trees (with N. Matte Bon).
  Trans. Amer. Math. Soc., Volume 376, Issue 10, 2023, pp. 7187–7233.
- Subgroup dynamics and C^*-simplicity of groups of homeomorphisms (with N. Matte Bon).
  Ann. Sci. École Norm. Sup., 51, fascicule 3 (2018), 557-602.

Chapter 2:

- Growth of actions of solvable groups (with N. Matte Bon).
  https://arxiv.org/abs/2205.11924.

Chapter 3:

- Bounding the covolume of lattices in products (with P.-E. Caprace).
  Compos. Math., Vol. 155, Issue 12, 2019, pp. 2296-2333.

Chapter 4:

- Commensurated subgroups and micro-supported actions (with P.-E. Caprace). Appendix by D. Francoeur.
  J. Eur. Math. Soc. (JEMS), Volume 25, No 6, 2023, pp. 2251–2294.

Chapter 5:

- Rigidity and flexibility results for groups with a common cocompact envelope.
  https://arxiv.org/abs/2510.24581.

Chapter 6:

- Commability and envelopes of Baumslag-Solitar groups and generalizations (with Y. Cornulier). Preprint.

Other works (2015-2025)

- On commensurators of free groups and free pro-p groups (with Y. Barnea, M. Ershov, C. Reid, M. Vannacci, T. Weigel). https://arxiv.org/abs/2507.04120.
- On closure operations in the space of subgroups and applications (with D. Francoeur).
  Ergodic Theory Dynam. Systems, Volume 45, Issue 12, 2025, pp. 3728-3748.
- On the growth of actions of free products (with N. Matte Bon and V. Salo).
  Groups Geom. Dyn., Vol. 19, Issue 2 (2025), 661–680.
- Continuity of the stabilizer map and irreducible extensions (with T. Tsankov).
  Comment. Math. Helv., Vol. 100, Issue 1 (2025), 123–146.
- Some torsion-free solvable groups with few subquotients (with N. Matte Bon).
  Math. Proc. Cambridge Philos. Soc., Vol. 176, Issue 2 (2024), 279–286.
- Confined subgroups and high transitivity (with N. Matte Bon).
  Ann. H. Lebesgue, Volume 5 (2022), pp. 491-522.
- Triple transitivity and non-free actions in dimension one (with N. Matte Bon).
  J. Lond. Math. Soc., Volume 105, Issue 2, 2022, pp. 884-908.
- Amenable uniformly recurrent subgroups and lattice embeddings.
  Ergodic Theory Dynam. Systems, Volume 41, Issue 5, 2021, pp. 1464-1501.
- Simple groups and irreducible lattices in wreath products.
  Ergodic Theory Dynam. Systems, Volume 41, Issue 5, 2021, pp. 1502-1513.
- Locally compact groups whose ergodic or minimal actions are all free (with N. Matte Bon).
  Int. Math. Res. Not., Vol. 2020, Issue 11, June 2020, 3318–3340.
- Densely related groups (with Y. de Cornulier).
  Ann. Fac. Sci. Toulouse Math., Série 6, Tome 28 (2019) no. 4, p. 619-653.
- Commensurated subgroups in tree almost automorphism groups (with Ph. Wesolek).
  Groups Geom. Dyn.,Vol. 13, Issue 1 (2019), 1-30.

Other works (until 2015)

- C^*-simplicity and the amenable radical.
  Invent. Math., Vol. 209, Issue 1 (2017), 159-174.
- Groups acting on trees with almost prescribed local action.
  Comment. Math. Helv., Vol. 91, Issue 2 (2016), 253-293.
- Locally compact lacunary hyperbolic groups.
  Groups Geom. Dyn., Vol. 11 Issue 2 (2017), 415-454.
- Compact presentability of tree almost automorphism groups.
  Ann. Inst. Fourier, Vol. 67 no. 1 (2017), 329-365.
- The divergence of the special linear group over a function ring.
  Comm. Algebra, Vol 43, No 9 (2015), 3636-3654.

CHAPTER 1

Subgroup dynamics, topological dynamics of non-free actions

In this chapter we present results from [**LBMB18, LBMB23**], joint work with Nicolás Matte Bon, and from [**CLBMB22**], joint work with Pierre-Emmanuel Caprace and Nicolás Matte Bon. The reader is invited to consult Appendix A and Appendix B before reading the present chapter.

1. Uniformly recurrent subgroups, confined subgroups

Let G be a locally compact group. We denote by $\mathrm{Sub}(G)$ the space of closed subgroups of G. Equipped with the Chabauty topology, $\mathrm{Sub}(G)$ is a compact space. The group G acts by conjugation on $\mathrm{Sub}(G)$.

Definition 1.1 ([**GW15**]). A **uniformly recurrent subgroup** (URS) of G is a minimal closed non-empty G-invariant subset of $\mathrm{Sub}(G)$.

URS's appear when considering point stabilizers for minimal actions on compact spaces. Let X be a minimal compact G-space. For $x \in X$, let G_x denote the stabilizer of x. The G-equivariant map $\mathrm{Stab} : X \to \mathrm{Sub}(G)$, $x \mapsto G_x$, is called the stabilizer map. The map Stab is not continuous in general. When G is a discrete group, the points where Stab is continuous are the points $x \in X$ such that $G_x = G_x^0$, where G_x^0 is the neighbourhood stabilizer of x in G (§1.1 in Appendix A). See e.g. [**Nek22**, Proposition 2.1.28].

However the map Stab is always upper semi-continuous. Proposition A.1 therefore yields:

Proposition 1.2 (Stabilizer URS, [**GW15**]). *If X is a minimal compact G-space, then*

$$\overline{\mathrm{Stab}(X)} = \overline{\{G_x \,:\, x \in X\}}$$

contains a unique URS. This URS is denoted $S_G(X)$, and is called the stabilizer URS associated to X. When G is second countable, the subset $X_0 \subset X$ consisting of points where Stab *is continuous is dense in X, and*

$$S_G(X) = \overline{\{G_x \,:\, x \in X_0\}}.$$

So the proposition associates a URS of G to every minimal compact G-space. Conversely, Matte Bon and Tsankov, and Elek in the case of finitely generated discrete groups, showed that every URS of G can be obtained as the stabilizer URS associated to a minimal compact G-space [**MBT20**], [**Ele18**].

The action of G on X is **topologically free** if $S_G(X)$ is the trivial URS. By the trivial URS we mean the singleton $\{\{1\}\}$ made of the trivial subgroup. For G second countable, this amounts to say that there is a dense set of points in X with trivial stabilizer. When the group G is discrete, the action is topologically free if and only if

G_x^0 is trivial for every $x \in X$ (equivalently, the only element of G whose set of fixed points in X has non-empty interior is the identity).

Closely related to the notion of URS is the notion of confined subgroups:

Definition 1.3. A closed subgroup H of G is **confined** if the closure of the G-orbit of H in $\mathrm{Sub}(G)$ does not contain the trivial subgroup.

The sets $\{H \in \mathrm{Sub}(G) \mid H \cap K = \emptyset\}$, when K ranges over compact subsets of G such that $1 \notin K$, form a basis of neighbourhoods of the trivial subgroup in $\mathrm{Sub}(G)$. Hence H is confined means there exists a compact subset K such that $1 \notin K$ and $K \cap gHg^{-1} \neq \emptyset$ for every $g \in G$. Note that any subgroup containing a confined subgroup is itself confined. If H is confined, by Zorn's lemma the closure of the G-orbit of H in $\mathrm{Sub}(G)$ contains a URS, that is necessarily non-trivial. Conversely if $\mathcal{H} \subset \mathrm{Sub}(G)$ is a non-trivial URS, then every $H \in \mathcal{H}$ is confined.

Everywhere in Chapter 1 except in Section 5, the group G will be discrete. In that case a subgroup H is confined if there exists a *finite* subset P of non-trivial elements of G such that $gHg^{-1} \cap P \neq \emptyset$ for every $g \in G$. When G is a finitely generated group, this admits an interpretation in terms of Schreier graphs. If S is a finite generating subset of G, the Schreier graph $\Gamma(G, G/H)$ associated to a subgroup H of G is the graph with vertex set G/H, and edges between gH and sgH for every $s \in S$, $g \in G$. A subgroup H of G is *not* confined if and only if the Schreier graph $\Gamma(G, G/H)$ contains isomorphic copies (as labelled graphs) of arbitrarily large balls of the Cayley graph of G (all graphs being taken with respect to S).

Outline of Chapter 1. One aspect of the study of confined subgroups and URS's is to aim towards rigidity results describing all confined subgroups or URS's of G. In a series of works, joint with Nicolás Matte Bon, we have established such results, and developed applications of these, for discrete groups admitting a micro-supported action [**LBMB18, LBMB20, LBMB23, LBMB22a**]. The purpose of Sections 2, 3 and 4 is to describe results from [**LBMB18, LBMB23**]. Section 5 describes results from [**CLBMB22**], which deals with the study of relatively amenable confined subgroups of non-discrete locally compact micro-supported groups.

2. Confined subgroups and URS's of micro-supported groups

The following statement makes a connection, given a micro-supported group, between the rigid stabilizers of the given micro-supported action and all confined subgroups of G.

Theorem 1.4 ([**LBMB18, LBMB23**]). *Let G be a discrete group admitting a micro-supported action on a Hausdorff space X. If H is a confined subgroup of G, then there exists a non-empty open subset U of X such that H contains* $\mathrm{Rist}_G(U)'$.

We make somes comments:

- Lemma B.2 asserts that if G is a group admitting a micro-supported action of a Hausdorff space X and N is a non-trivial normal subgroup of G, then N contains $\mathrm{Rist}_G(U)'$ for some non-empty open subset $U \subset X$. Hence Theorem 1.4 can be seen as an extension of this basic fact on normal subgroups to the setting of confined subgroups.

- A version of Theorem 1.4 was first obtained in [**LBMB18**], with the weaker conclusion that $\mathrm{Rist}_G(U)'$ is a subquotient of H. The above stronger statement was then later obtained in [**LBMB23**].
- In the setting of Invariant Random Subgroups (G-invariant probability measures on $\mathrm{Sub}(G)$) – a topic on which we do not attempt to make a survey here – the exact analogue of Theorem 1.4 was obtained by Zheng [**Zhe19a**]. Previous results on IRS's for specific families of micro-supported groups had been obtained notably by Dudko–Medynets and Thomas–Tucker-Drob. We refer to [**Zhe19a**] for details and references.
- Theorem 1.4 is actually derived from a more constructive statement (Theorem 3.21 in [**LBMB23**]), which has the advantage of being applicable outside of the realm of micro-supported groups, and has other applications. In the work [**LBMB22a**], we apply this result to a problem that admits a priori no connection with confined subgroups, namely the study of highly transitive actions. As an output, we obtain a criterion to rule out the existence of highly transitive actions, as well as a tool to classify highly transitive actions.
- Another application of (a consequence of) Theorem 1.4 is derived and used as tool in the main result of [**LBMB20**].

Theorem 1.4 is therefore a tool to study confined subgroups and URS's of micro-supported groups. Under additional assumptions on the G-action on X, the following statement provides a criterion ensuring that the only URS of G, except $\{\{1\}\}$ and $\{G\}$, is the natural one, i.e. the stabilizer URS (Proposition 1.2) associated to the minimal micro-supported action from which G is given.

THEOREM 1.5 ([**LBMB18**]). *Let G be a discrete group admitting a micro-supported action on a compact space X that is minimal and extremely proximal. Assume that:*

(i) There is a basis for the topology consisting of open subsets U of X such that the rigid stabilizer $\mathrm{Rist}_G(U)$ is perfect.
(ii) The subgroups G_x^0, where x ranges over X, generate G.

Then the only URS's of G are $\{\{1\}\}$, $\{G\}$ and the stabilizer URS $S_G(X)$ associated to X.

REMARK 1.6. Theorem 1.5 is proven in [**LBMB18**] under the assumption that there exists $H \in S_G(X)$ such that H is a maximal subgroup of G. However by [**LB21**, Proposition 3.17] this assumption can be replaced by the above assumption (ii).

Apart from specific situations where the group G is too small (for instance when $\mathrm{Sub}(G)$ is countable), situations where we have a complete description of all URS's of G are rather rare, and even more if we stick to finitely generated groups G. Recent results by Boutonnet–Houdayer [**BH21**] and Bader–Gelander–Levit [**BGL24**] solve this problem for irreducible lattices in connected semisimple Lie groups with trivial center and rank at least two: for such a group, every non-trivial URS (and more generally every confined subgroup) is the conjugacy class of a finite index subgroup.

As an illustration, we obtain the following statement for Thompson's groups F, T and V. For the groups T and V, this a direct application of Theorem 1.5, as their defining micro-supported action can be shown to verify all requirements of Theorem 1.5. The case of the group F is slightly different, as the defining micro-supported action of F (on the unit interval) is not minimal.

THEOREM 1.7 ([**LBMB18**]).

(i) The only URS's of Thompson's group F are the normal subgroups of F (these are the subgroups containing the derived subgroup F').
(ii) The only URS's of Thompson's group T are $\{\{1\}\}$, $\{T\}$, and the stabilizer URS associated to the action of T on the circle.
(iii) The only URS's of Thompson's group V are $\{\{1\}\}$, $\{V\}$, and the stabilizer URS associated to the action of V on the binary Cantor space.

As another illustration, Theorem 1.5 also applies to the family of groups $G(F, F')$ acting on a tree T_d with almost prescribed local action. Under certain assumptions on the permutation groups F, F', the micro-supported action of $G(F, F')$ on ∂T_d verifies the assumptions of Theorem 1.5. Examples of F, F' satisfying the conditions below are $F = \langle(1, \dots, d)\rangle$ and $F' = \mathrm{Alt}(d)$ for $d \geq 7$ odd.

THEOREM 1.8 ([**LBMB18**]). *Let $d \geq 3$, and let $F \leq F' \leq \mathrm{Sym}(d)$ such that F acts freely transitively, F' acts 2-transitively, and point stabilizers in F' are perfect. Let G be the subgroup of index two of $G(F, F')$ preserving the natural bipartition of the vertex set of the regular tree T_d. Then the only URS's of G are $\{\{1\}\}$, $\{G\}$ and the stabilizer URS associated to the action of G on ∂T_d.*

3. Rigidity results for non topologically free actions

3.1. Extremely proximal micro-supported groups. Suppose the group G is as in Theorem 1.5, and let be Y a minimal compact G-space. The stabilizer URS $S_G(Y)$ is therefore equal to either $\{\{1\}\}$, $\{G\}$ or $S_G(X)$. The first case corresponds to topologically free actions. Only the trivial G-space gives rise to the second case. Hence all the non-trivial minimal compact G-spaces Y such that the G-action on Y is not topologically free verify $S_G(Y) = S_G(X)$. Using this one shows the following statement (whose conclusion is at the level of the G-spaces rather than the stabilizer URS's).

THEOREM 1.9 ([**LBMB18, LBMB23**]). *Let G be a discrete group admitting a micro-supported action on a compact space X that is minimal and extremely proximal. Assume that:*

(i) There is a basis for the topology consisting of open subsets U of X such that the rigid stabilizer $\mathrm{Rist}_G(U)$ is perfect.
(ii) For every $x_1 \neq x_2 \in X$, the subgroups $G^0_{x_1}$ and $G^0_{x_2}$ generate G.

Let Y a non-trivial minimal compact G-space such that the G-action on Y is not topologically free. Then Y admits X as a factor.

If G admits a minimal micro-supported action on a compact space X, then all compact G-spaces on which the G-action is micro-supported admit a common highly proximal extension (the Stone space of the Boolean algebra of regular open subsets of X) [**Rub96**]. The difference here is that Theorem 1.9 applies to all minimal compact G-spaces on which the G-action is not topologically free, and not only those that are micro-supported. But already if we stick to micro-supported G-spaces, the above theorem provides a kind of complement: while [**Rub96**] ensures that there always exists a largest micro-supported G-space, Theorem 1.9 provides sufficient conditions ensuring the existence of a smallest one.

This statement applies for instance to Thompson's groups T and V, as well as to the groups $G(F, F')$ when the parameters F, F' verify the assumptions of Theorem 1.8.

3.2. Weakly branch groups. Another situation in which we are able to say something on all non topologically free actions, based on a result on URS's, is the case of weakly branch groups. The set of URS's of a weakly branch group never admits a simple description as in Theorem 1.5. Corollary 5.10 in [**LBMB23**] asserts that every weakly branch group admits uncountably many distinct URS's. Nevertheless, a structure theorem on URS's of weakly branch groups is obtained in [**LBMB23**], which is again based on Theorem 1.4. That statement is a bit technical to be stated here, but it has the following consequence. (If T is a locally finite rooted tree, when saying that a subgroup $G \leq \mathrm{Aut}(T)$ is a weakly branch group we mean that the G-action on ∂T is minimal and micro-supported.)

THEOREM 1.10 ([**LBMB23**]). *Let T be a locally finite rooted tree, and let $G \leq \mathrm{Aut}(T)$ be a weakly branch group. Let Y a minimal compact G-space such that the G-action on Y is faithful and not topologically free. Then the following hold:*

(1) For every $y \in Y$, the subgroup G_y^0 admits fixed points in ∂T.
(2) The map $\Psi\colon Y \to \mathcal{F}(\partial T)$ which associates to $y \in Y$ the set of fixed points in ∂T of G_y^0, is continuous, G-equivariant, and the image of Ψ has cardinality at least two.
(3) If G is finitely generated and the growth of the G-action on Y is polynomially bounded, then the image of Ψ is infinite. (We refer to Section 1 of Chapter 2 for the definition of the growth of a G-action.)

The space $\mathcal{F}(\partial T)$ of closed subsets of ∂T, equipped with the Chabauty topology, is a compact space, and the fact that the G-action on ∂T is profinite implies that the G-action on $\mathcal{F}(\partial T)$ is also profinite ([**LBMB23**, Lemma 5.8]). Hence 2 implies in particular that Y admits a non-trivial profinite G-space as a factor, and 3 implies that Y admits an infinite profinite G-space as a factor.

Theorem 1.10 implies the following. To put into context, it is known that for every countable group G with trivial FC-center, there exists a minimal compact G-space on which the G-action is proximal and topologically free [**GTWZ21**].

COROLLARY 1.11 ([**LBMB23**]). *Let G be a weakly branch group. Then every minimal compact G-space on which the G-action is faithful and proximal is topologically free.*

Conclusion 3 of Theorem 1.10 is used to show the following. See [**Dah19**] for background on the group of interval exchange transformations.

THEOREM 1.12 ([**LBMB23**]). *If G is a finitely generated weakly branch group, then G does not admit any faithful action on $\mathbb{R}/\mathbb{Z}$ by interval exchange transformations.*

Rigidity results on confined subgroups and URS's as in Section 2 also lead to lower bounds for the growth of all faithful actions of the group (a problem studied in Chapter 2 for a different class of groups). See [**MB18**] for topological full groups, and [**LBMB23**] for branch groups, as well as [**LBMBS25**] for free products.

4. Amenable confined subgroups, G-boundaries, C^*-simplicity

A group G is **C^*-simple** if its reduced C^*-algebra is simple. This property can be rephrased in terms of unitary representations: G is C^*-simple if and only if every unitary representation of G that is weakly contained in the left-regular representation λ_G is weakly equivalent to λ_G [**Har07**].

Theorem 1.13 ([**KK17, Ken20**]). *Let G be a discrete group. The following conditions are equivalent:*

(i) The G-action on G on its Furstenberg boundary $\partial_F G$ is free.
(ii) There is a G-boundary on which the G-action is topologically free.
(iii) G does not have any amenable confined subgroup.
(iv) The only amenable URS of G is the trivial URS.
(v) G is C^*-simple.

Equivalence between conditions (i)-(ii)-(iii)-(iv) goes as follows. (i) $\Longrightarrow$ (ii) is tautological. For (ii) $\Longrightarrow$ (iii), suppose for a contradiction that X is a G-boundary on which the G-action is topologically free, and H is an amenable confined subgroup of G. Since H is amenable, there is $\mu \in \mathrm{Prob}(X)$ that is fixed by H. By strong proximality, there is a net (g_i) and $x \in X$ such that $g_i\mu \to \delta_x$. By compactness one can assume that there is $K \in \mathrm{Sub}(G)$ such that $g_i H g_i^{-1} \to K$, and by upper semi-continuity of Stab we have that K fixes x. Since H is confined, so is K. And since $K \leq G_x$, so is G_x. Therefore the URS $S_G(X)$ is non-trivial. Contradiction. (iii) $\Longrightarrow$ (iv) is consequence of definitions. (iv) $\Longrightarrow$ (i): point stabilizers for the G-action on $\partial_F G$ are always amenable, hence (iv) implies that the G-action on $\partial_F G$ is topologically free, and as observed in [**BKKO17**], it is a consequence of [**Fro68**] that this happens only if the G-action on $\partial_F G$ is free.

The remarkable aspect of Theorem 1.13 is that those conditions (i)-(ii)-(iii)-(iv) are also equivalent to (v). This has been originally proven by Kalantar–Kennedy [**KK17**], establishing equivalence between (ii) and (v). The connection with amenable confined subgroups has been made later by Kennedy [**Ken20**]. A subsequent result of Breuillard–Kalantar–Kennedy–Ozawa characterizes groups with the unique trace property as those with no non-trivial amenable *normal* subgroup [**BKKO17**]. By [**LB17**], this class of groups properly contains the class of groups satisfying the equivalent properties of Theorem 1.13. Theorem 1.13 has also been given another proof in [**BKKO17**], and has been exploited there to provide new proofs of C^*-simplicity of many previously known examples. Prior to [**KK17, BKKO17**], almost all known proofs of C^*-simplicity were implicitly or explicitly based on the existence of an action (most of the time isometric) satisfying a weak form of properness as well as a weak form of non-positive curvature. Examples of such situations covered non-abelian free groups [**Pow75**], non-trivial free products [**PS79**], hyperbolic groups [**Har88**], relatively hyperbolic groups [**AM07**], lattices in semisimple connected Lie groups [**BCH94**], centerless mapping class groups and outer automorphism groups of free groups [**BH04**], acylindrically hyperbolic groups [**DGO17**].

The following statement is a direct consequence of the combination of Theorem 1.13 and Theorem 1.4. It provides a criterion to show C^*-simplicity among the class of micro-supported groups (a class of groups that is disjoint from all the above classes of groups).

Corollary 1.14 ([**LBMB18**]). *Let G be a discrete group admitting a micro-supported action on a Hausdorff space X such that the rigid stabilizer of every non-empty open subset of X is non-amenable. Then G is C^*-simple.*

Corollary 1.15 ([**LBMB18**]). *The following groups are C^*-simple:*

(1) Thompson's group V; as well as the higher-dimensional nV, $n \geq 2$.
(2) Any non-amenable branch group.
(3) Any topological full group $\mathbb{F}(\Lambda, X)$ associated to a free minimal action of a non-amenable group Λ on the Cantor space X.
(4) the groups $H(A)$ and $G(A)$ of piecewise projective homeomorphisms of $\mathbb{R}$ and $\mathbb{P}^1(\mathbb{R})$ associated to a dense subring A of $\mathbb{R}$.
(5) Any group $\mathrm{P}(G)$ of piecewise prescribed automorphisms of a tree T associated to a subgroup G of $\mathrm{Aut}(T)$ such that the action of G on T is minimal and of general type and with non-amenable edge stabilizers.

5. Non-discrete locally compact micro-supported groups

This section extends the discussion of Section 4 to the situation where the group G is no longer discrete. This section is the only part of Chapter 1 that does not rely on Theorem 1.4.

When G is a locally compact group, the equivalence between properties (i)-(ii)-(iii)-(iv) from Theorem 1.13 remain true provided "amenable" is replaced by "relatively amenable". We refer to [**CM14**] for the notion of relative amenability. The proof goes as in the discussion below Theorem 1.13. However the equivalence between freeness and topological freeness of the action on $\partial_F G$ no longer follows from [**Fro68**]. It is nevertheless true, and follows from the main result of [**LBT25**] (joint work with Todor Tsankov). See Corollary 1.3 in [**LBT25**].

While, as seen in Section 4, many familiar discrete groups have no amenable confined subgroups, many classical examples of non-discrete locally compact groups do have amenable confined subgroups. This is notably the case for semisimple Lie groups and semisimple algebraic groups over local fields, or for any closed subgroup $G \leq \mathrm{Aut}(T_d)$ of the automorphism group of a d-regular tree T_d ($d \geq 3$) acting 2-transitively on ∂T_d: any such group G has a cocompact (and hence confined) amenable subgroup. Additional examples include closed subgroups $G \leq \mathrm{Aut}(\Delta)$ of the automorphism group a locally finite building acting strongly transitively on Δ [**CL11**, Theorem 4.10].

The following result provides a sufficient condition ensuring that a locally compact micro-supported group admits no relatively amenable confined subgroup. The criterion applies under a stronger requirement than micro-supported. If G is a totally disconnected locally compact (tdlc) group and X a totally disconnected compact G-space, we say that the G-action on X is **piecewise minimal-strongly-proximal** if for every non-empty clopen subset U of X, the action of the rigid stabilizer $\mathrm{Rist}_G(U)$ on U is minimal and strongly proximal.

Theorem 1.16 ([**CLBMB22**]). *Let G be a tdlc group admitting a faithful and piecewise minimal-strongly-proximal action. Then G does not have any relatively amenable confined subgroup.*

Note that the definition of piecewise minimal-strongly-proximal implies in particular that for every non-empty clopen subset U of X, the rigid stabilizer $\mathrm{Rist}_G(U)$ is non-amenable. Hence *discrete groups* as in Theorem 1.16 are already known not to have amenable confined subgroups in view of Theorem 1.4.

Following [**CRW17b**], let $\mathcal{S}$ denote the class of non-discrete tdlc groups that are compactly generated and topologically simple. We refer to [**Cap18, CW23**] for recent results highlighting the role of the class $\mathcal{S}$ in the study of tdlc groups. Results from [**CRW17b**, § 1.4, § 6] split the class $\mathcal{S}$ into four disjoint subclasses, one of which being the class of micro-supported groups in $\mathcal{S}$. Here in order to simplify the discussion, we restrict to the case of abstractly simple groups G in $\mathcal{S}$. For those, exactly one of the following three situations occurs:

- (Locally hereditarily just-infinite) Every compact open subgroup U of G is just-infinite. Just-infinite means U is infinite and every proper quotient of U is finite.
- (Micro-supported) There is a compact totally disconnected G-space on which the action is micro-supported, minimal and strongly proximal.
- (Non-principal filter type) These are characterized by the property that G has an infinite non-open compact locally normal subgroup, and no two infinite compact locally normal subgroups commute. Locally normal means that the subgroup has open normalizer.

In the micro-supported situation, it is not always the case that the action is piecewise minimal-strongly-proximal. For instance, consider the index two subgroup G of the automorphism group $\mathrm{Aut}(T_d)$ of a d-regular tree T_d. The group G is non-discrete, compactly generated, and abstractly simple by [**Tit70**]. The G-action on ∂T_d is micro-supported, but the rigid stabilizer $\mathrm{Rist}_G(U)$ of a non-empty proper clopen subset U of ∂T_d is a compact subgroup of G, and acts on U with an invariant probability measure.

There are nevertheless various examples of micro-supported groups in $\mathcal{S}$ to which Theorem 1.16 applies. Examples are the Neretin groups:

Corollary 1.17 ([**CLBMB22**]). *For every $d, k \geq 2$, the Neretin group $\mathcal{N}_{d,k}$ does not have any relatively amenable confined subgroup.*

One property of the group $\mathcal{N}_{d,k}$ that is deduced from Corollary 1.17 in [**CLBMB22**] is that $\mathcal{N}_{d,k}$ admits (explicit) weakly equivalent quasi-regular irreducible representations that are not equivalent. This property subsequently gave rise to further work of Arimoto [**Ari22**]. More recently, in a related direction Morando showed that the regular representation of $\mathcal{N}_{d,k}$ is factorial [**Mor25**].

6. Other works (directly or indirectly) related to Chapter 1

(1) On closure operations in the space of subgroups and applications (with D. Francoeur).
Ergodic Theory Dynam. Systems, Volume 45, Issue 12, 2025, pp. 3728-3748.
(2) On the growth of actions of free products (with N. Matte Bon and V. Salo).
Groups Geom. Dyn., Vol. 19, Issue 2 (2025), 661–680.
(3) Continuity of the stabilizer map and irreducible extensions (with T. Tsankov).
Comment. Math. Helv., Vol. 100, Issue 1 (2025), 123–146.
(4) Confined subgroups and high transitivity (with N. Matte Bon).
Ann. H. Lebesgue, Volume 5 (2022), pp. 491-522.

(5) Amenable uniformly recurrent subgroups and lattice embeddings.
Ergodic Theory Dynam. Systems, Volume 41, Issue 5, 2021, pp. 1464-1501.
(6) Locally compact groups whose ergodic or minimal actions are all free (with N. Matte Bon).
Int. Math. Res. Not., Vol. 2020, Issue 11, June 2020, 3318–3340.

CHAPTER 2

On the geometry of Schreier graphs of solvable groups

In this chapter we present results from [**LBMB22b**], joint work with Nicolás Matte Bon.

1. Growth of Schreier graphs

Let G be a finitely generated group, and S a finite symmetric generating subset. If X is a G-set, we denote by $\Gamma(G, X)$ the graph whose vertex set is X, and for every $x \in X$ and $s \in S$ there is an edge connecting x to sx. The graph $\Gamma(G, X)$ is called the **Schreier graph** of the action of G on X. The group G being fixed, we are interested in common geometric properties of all Schreier graphs of faithful actions of G. In the sequel we focus on their growth. The **growth of the action** of G on X is the function that measures the maximal cardinality of a ball of radius n in $\Gamma(G, X)$:

$$\mathrm{vol}_{G,X}(n) = \max_{x \in X} |S^n \cdot x|.$$

Given functions $f, g \colon \mathbb{N} \to \mathbb{N}$, we write $f(n) \preccurlyeq g(n)$ if there is a constant $C > 0$ such that $f(n) \leq Cg(Cn)$, and $f(n) \simeq g(n)$ if $f(n) \preccurlyeq g(n)$ and $g(n) \preccurlyeq f(n)$. The function $\mathrm{vol}_{G,X}(n)$ does not depend on the choice of S up to $\simeq$.

For the left translation action of G on itself, $\mathrm{vol}_{G,X}(n)$ is the classical growth of the group G, denoted $\mathrm{vol}_G(n)$. It is clear that for every G-set X we have $\mathrm{vol}_{G,X}(n) \preccurlyeq \mathrm{vol}_G(n)$. Various groups admit faithful actions for which the function $\mathrm{vol}_{G,X}(n)$ is strictly smaller than $\mathrm{vol}_G(n)$. Classical instances of groups naturally coming with a faithful action such that $\mathrm{vol}_{G,X}(n) \simeq n$ are B.H. Neumann's examples of continuously many non-isomorphic finitely generated groups [**Neu37**], the Houghton groups, the Grigorchuk group [**BG00**], the lamplighter group $A \wr \mathbb{Z}$ for every finite group A, the topological full group of a $\mathbb{Z}$-action on the Cantor space, and Nekrashevych's groups obtained as fragmentation of dihedral groups [**Nek18**]. Actions of linear growth, and more generally the analysis of graphs of actions, played a crucial role in the recent developments on topological full groups and other related groups [**JM13, JNS16, Nek18, BNZ22**]. Other examples of groups admitting faithful actions of linear growth are virtually abelian groups, non-abelian free groups [**Sch27**], and right-angled Artin groups [**Sal21**].

We are interested in understanding obstructions to the existence of faithful actions of small growth. We introduce the following definition:

Definition 2.1. Let $f \colon \mathbb{N} \to \mathbb{R}_+$. A finitely generated group G has a **Schreier growth gap** $f(n)$ if every faithful G-set X satisfies $\mathrm{vol}_{G,X}(n) \succcurlyeq f(n)$.

We always have $\mathrm{vol}_{G,X}(n) \succcurlyeq n$ for every faithful G-set X provided the group G is infinite. We are interested in Schreier growth gaps where the function f is super-linear ($f(n)/n$ is unbounded).

If a group G has a Schreier growth gap $f(n)$, then the same is true for any group having G as a subgroup. Hence in the study of Schreier growth gaps it is natural consider groups that are generally thought of being rather small. In the work [**LBMB22b**] we carry out a study of Schreier growth gaps for finitely generated solvable groups.

2. Metabelian groups

As mentioned above, every virtually abelian group admits a faithful action with linear growth. Hence in the sequel virtually abelian groups are permanently excluded from the discussion.

The situation for metabelian groups is already diverse. The starting observation is that there are metabelian groups of exponential growth which admit faithful actions of polynomial growth. Examples are the wreath product $G = A \wr B$ of two finitely generated abelian groups. The group G is the semi-direct product $\oplus_B A \rtimes B$, where $\oplus_B A$ is the set of finitely supported functions $B \to A$, and B acts on $\oplus_B A$ by $b \cdot \varphi : b' \mapsto \varphi(b^{-1}b')$. The action of G on the set $B \times A$ defined by $(\varphi, b) \cdot (b_0, a_0) = (bb_0, \varphi(bb_0)a_0)$ is called the standard wreath product action. With natural choices of generators, the graph of this action is obtained by taking a copy of the Cayley graph of A and attaching to each vertex a copy of the Cayley graph of B, and its growth is equivalent to the growth of the abelian group $B \times A$. So for the lamplighter group $G = C_p \wr \mathbb{Z}$ (where C_p is the cyclic group of order p), the standard wreath product action satisfies $\mathrm{vol}_{G,X}(n) \simeq n$. Hence in particular there is no Schreier growth gap that is uniform for all (non-virtually abelian) metabelian groups. Nevertheless we establish a uniform quadratic gap in the following two situations:

Theorem 2.2 ([**LBMB22b**]). *Let G be a finitely generated metabelian group that is not virtually abelian. Suppose that G satisfies at least one of the following:*

i) G is finitely presented;
ii) G is torsion-free.

Then G has a Schreier growth gap n^2.

The following result establishes a more quantitative estimate. Whenever G is a finitely generated metabelian group and $1 \to M \to G \to Q \to 1$ is a short exact sequence with M, Q abelian, M can be seen as a finitely generated module over the group ring $\mathbb{Z}Q$. This point of view plays a crucial role in the study of metabelian groups since the seminal work of Hall [**Hal54**]. The Krull dimension of the $\mathbb{Z}Q$-module M does not depend on the choice of (M, Q) provided G is not virtually abelian. This positive integer is called the Krull dimension of G [**LS03**, §9.4], [**Cor11**], [**Jac19**].

Theorem 2.3 ([**LBMB22b**]). *Let G be a finitely generated metabelian group which is not virtually abelian, and let $k = \dim_{\mathrm{Krull}}(G)$. Then G has a Schreier growth gap n^k.*

We have $\dim_{\mathrm{Krull}}(C_p \wr \mathbb{Z}^d) = d$ and $\dim_{\mathrm{Krull}}(\mathbb{Z} \wr \mathbb{Z}^d) = d+1$, so the group $C_p \wr \mathbb{Z}^d$ has a Schreier growth gap n^d, and the group $\mathbb{Z} \wr \mathbb{Z}^d$ has a Schreier growth gap n^{d+1}. In both cases the estimate is sharp, as the bound is attained by the standard wreath product action. As another example, for the free metabelian group on $d \geq 2$ generators, we obtain a Schreier growth gap n^{d+1} (and this estimate is also sharp). The case of the free metabelian group is quite specific among free solvable groups, as we also show that the free solvable group of rank $d \geq 2$ and solvability length $\ell \geq 3$ has a Schreier growth gap $\exp(n)$ [**LBMB22b**, Theorem 7.22].

3. Solvable groups

We now consider solvable groups of higher solvability length, and first focus on the class of solvable groups of finite rank. Recall that a solvable group G is of finite rank (or finite Prüfer rank) if there is an integer k such that every finitely generated subgroup of G can be generated by at most k elements. A wreath product $A \wr B$ is never of finite rank provided A is non-trivial and B is infinite. Every polycyclic group is of finite rank. The solvable Baumslag–Solitar group $\mathbb{Z}[1/n] \rtimes_n \mathbb{Z}$, $n \geq 2$, is an example of a non-polycyclic group that is of finite rank. More generally, every finitely generated solvable group that is linear over $\mathbb{Q}$ is of finite rank (and those groups are precisely the finitely generated solvable groups of finite rank that are virtually torsion-free).

THEOREM 2.4 ([**LBMB22b**]). *Let G be a finitely generated solvable group of finite rank, and assume that G is not virtually nilpotent. Then G has a Schreier growth gap* $\exp(n)$.

Moving to arbitrary solvable groups, we conjecture that the quadratic Schreier growth gap for torsion-free metabelian groups from Theorem 2.2 extends to torsion-free solvable groups:

CONJECTURE 2.5 ([**LBMB22b**]). *Let G be a finitely generated solvable group which is virtually torsion-free. If G is not virtually abelian, then G has a Schreier growth gap n^2.*

The conjecture is true in the following situations:

THEOREM 2.6 ([**LBMB22b**]). *Let G be a finitely generated solvable group which is virtually torsion-free. Suppose that G satisfies at least one of the following:*

- *i) G admits a nilpotent normal subgroup N such that G/N is virtually abelian (e.g. G is linear);*
- *ii) the successive quotients in the Fitting series of G are torsion-free.*

Then Conjecture 2.5 is true for G.

Theorem 2.2 immediately implies that if a torsion-free group G contains a finitely generated metabelian subgroup which is not virtually abelian, then G has a Schreier growth gap n^2. It follows from classical arguments that this situation covers case i) of Theorem 2.6. The case ii) of Theorem 2.6 does not only rely on the case of metabelian groups. It involves a more technical mechanism that allows in some cases to lift the desired conclusion from a quotient to the ambient group.

While Theorem 2.6 establishes Conjecture 2.5 under additional assumptions on G, it is also natural to add assumptions on the actions. The following result asserts that the conjecture is true if we restrict to transitive actions, and more generally to actions with finitely many orbits.

THEOREM 2.7 ([**LBMB22b**]). *Let G be a finitely generated solvable group which is virtually torsion-free, and not virtually abelian. Let X be a faithful G-set such that the action of G on X has finitely many orbits. Then $\mathrm{vol}_{G,X}(n) \succcurlyeq n^2$.*

4. The method: non-foldable subsets and confined subgroups

Our approach to study Schreier growth gaps is based on the following notion of independent interest introduced in [**LBMB22b**].

DEFINITION 2.8. Let G be a group and $\mathcal{L}$ a subset of G.

- Let X be a G-set. We say $\mathcal{L}$ is **non-folded in** X if for every finite subset Σ of $\mathcal{L}$, there is $x \in X$ such that the orbital map $g \mapsto gx$ is injective on Σ.
- We say $\mathcal{L}$ is **non-foldable** if $\mathcal{L}$ is non-folded in X for every faithful G-set X.

Showing that a subset $\mathcal{L}$ of G is non-foldable consists in showing that certain subsets of $\mathrm{Sub}(G)$ made of confined subgroups of G are not faithful. The connection is as follows. If H is a confined subgroup of G and P is a finite subset of non-trivial elements of G such that $gHg^{-1} \cap P \neq \emptyset$ for every $g \in G$, then we say that P is a confining subset for H. We denote by $S_G(P, G)$ the set of confined subgroups H of G for which P is a confining subset. The subset $S_G(P, G)$ is closed and G-invariant in $\mathrm{Sub}(G)$. We say that a G-invariant subset $\mathcal{R} \subset \mathrm{Sub}(G)$ is not faithful if there is a non-trivial normal subgroup N of G such that $N \leq H$ for every $H \in \mathcal{R}$. Then Lemma 1.12 in [**LBMB22b**] asserts that a subset $\mathcal{L}$ of G is non-foldable if and only if for every finite subset $\Sigma \subset \mathcal{L}$, the set $S_G(P, G)$ is not faithful, where $P = \{g^{-1}h \colon g, h \in \Sigma, g \neq h\}$.

The data of a subset $\mathcal{L}$ of G that is non-folded in X provides information on the geometry of the graph $\Gamma(G, X)$, as by definition the graph $\Gamma(G, X)$ contains copies of arbitrarily large finite subsets of $\mathcal{L}$. In terms of growth, this implies that $\mathrm{vol}_{G,X}(n)$ must be at least equal to the relative growth $f_{(G,\mathcal{L})}(n)$ of $\mathcal{L}$ in G (the maximal cardinality of a ball of radius n in $\mathcal{L}$, where $\mathcal{L}$ is equipped with the induced metric from G). In particular if $\mathcal{L}$ is a non-foldable subset of G, then G has a Schreier growth gap $f_{(G,\mathcal{L})}(n)$. Our method to establish the results stated in Sections 2 and 3 consists in exhibiting non-foldable subsets (via a study of certain subsets $S_G(P, G)$ of confined subgroups as indicated above) that are sufficiently large, and explicit enough so that we can compute (or at least bound from below) their relative growth in G. We refer to [**LBMB22b**, Theorem 4.9] (which treats the case of solvable groups of finite rank of Theorem 2.4), to [**LBMB22b**, Theorem 6.10] (about metabelian groups; see also Theorems 6.24 and 6.27 respectively for torsion-free metabelian groups and finitely presented metabelian groups) for statements exhibiting non-foldable subsets.

CHAPTER 3

Lattices in products of simple locally compact groups

In this chapter we present some results from [**CLB19**], which is joint work with Pierre-Emmanuel Caprace.

1. The setting

The study of lattices Γ in semisimple Lie groups or algebraic groups over a local field gave rise to a tremendous amount of works. Some accounts of the developments of the theory that occurred between $\sim 1970-1990$ are recorded in the books [**Mar91, Zim84**]. The study of lattices $\Gamma \leq G$ in a locally compact group G that splits as a direct product $G = G_1 \times \cdots \times G_n$, $n \geq 2$, takes its origins from this world. In that setting the natural assumption that prevents Γ from being itself a direct product of lattices in each factor is to require that Γ has a dense projection on each factor.

Beyond the classical picture of semisimple or algebraic groups, the study of lattices in a locally compact group G that decomposes as a product $G = G_1 \times G_2$, where each G_i is a closed cocompact subgroup of the automorphism group of a locally finite regular tree, has been pioneered by Burger–Mozes. Among the achievements in [**BM97, BM00a, BM00b**], Burger–Mozes established an analogue of Margulis' normal subgroup theorem in this setting, which applies to cocompact lattices $\Gamma \leq G = G_1 \times G_2$ with dense projections and such that the factor groups G_1 and G_2 are certain topologically simple cocompact subgroups of the automorphism group of a regular tree (using notation introduced in Section 5 of Chapter 1, these factor groups are examples of groups in the class $\mathcal{S}$). This led to the discovery that cocompact lattices in product of automorphism groups of trees can enjoy drastically different properties than lattices in semisimple or algebraic groups. In particular Burger–Mozes exhibited cocompact lattices acting on a product of trees that are simple groups. Non-residually finite examples were also constructed by Wise [**Wis96**]. Those groups split as amalgamated product $\Gamma = F_1 *_{F_3} F_2$, where each F_i is a finitely generated non-abelian free group, and F_3 is embedded as a finite index subgroup in F_1 and F_2. A version of Margulis' normal subgroup theorem was then later established by Bader–Shalom in a high degree of generality [**BS06**]. We refer to the recent survey [**Cap19**] for an extensive discussion on this topic.

Rémy showed that Kac–Moody groups also provide a source of lattices in products of locally compact groups [**Ré99**]. In that setting Caprace–Rémy constructed finitely generated non-cocompact lattices in products $G = G_1 \times G_2$ of automorphism groups of buildings that are simple. Those are of different nature than lattices acting on product of trees, for instance because they have Kazhdan's property (T) [**CR09**].

In the work [**CLB19**] we consider cocompact lattices in products of compactly generated topologically simple locally compact groups, and establish results in this setting that are influenced by classical results in the setting of semisimple Lie groups.

Our results are actually valid in products of locally compact groups in which the factors satisfy a weaker condition than topological simplicity. We recall that a locally compact group G is called **just-non-compact** (JNC) if G is non-compact and every closed normal subgroup is trivial or cocompact. We say that G is **quasi just-non-compact** (QJNC) if G is non-compact and every closed normal subgroup is discrete or cocompact. See Section 3 below for the relevance of this property in the context of automorphism groups of connected locally finite graphs. Clearly JNC implies QJNC.

We note that a QJNC group is either totally disconnected, or almost connected (meaning that the identity component is cocompact). In the sequel we only consider the situation where the ambient group G is totally disconnected. Actually in the presence of a QJNC almost connected factor, the arithmeticity results of Caprace–Monod [**CM09**, Theorem 5.18] and Bader–Furman–Sauer [**BFS19**, Theorem 1.5] imply much stronger constraints on the lattices and on the other factors of G.

We use the shorthand tdlc for "totally disconnected locally compact".

2. Uniform discreteness

Let G be a semisimple Lie group without compact factor. A classical result of H. C. Wang asserts that for every lattice $\Gamma \leq G$, the collection of discrete subgroups of G containing Γ is finite [**Wan67**]. Bass–Kulkarni showed that in $G = \mathrm{Aut}(T_d)$, the automorphism group of the d-regular tree T_d, there exists infinite towers of cocompact lattices $\Gamma \subsetneq \Gamma_1 \cdots \subsetneq \Gamma_n \cdots$ [**BK90**, Theorem 7.1]. Since the group $\mathrm{Aut}(T_d)$ is compactly generated and has a simple open subgroup of index 2 [**Tit70**], Wang's theorem cannot be expected to hold for cocompact lattices in compactly generated topologically simple tdlc groups. The following is the analogue of Wang theorem for cocompact lattices with dense projections in products.

THEOREM 3.1 ([**CLB19**]). *Let $n \geq 2$, and let $G_1, \dots, G_n$ be non-discrete, compactly generated, QJNC tdlc groups. Let $\Gamma \leq G = G_1 \times \cdots \times G_n$ be a cocompact lattice such that the projection of Γ to G_i is dense in G_i for all i. Then the set of discrete subgroups of G containing Γ is finite.*

Given a compact subset $K \subseteq G$ of a locally compact group G, a subgroup H is called **K-cocompact** if $G = HK$.

THEOREM 3.2 ([**CLB19**]). *Let $n \geq 2$, let $G_1, \dots, G_n$ be non-discrete compactly generated QJNC tdlc groups, and let $G = G_1 \times \cdots \times G_n$. Let K be any compact subset of G, and let $\mathcal{L}_K(G)$ be the set of discrete K-cocompact subgroups $\Gamma \leq G$ with dense projection in G_i for all i. Then there exists an identity neighbourhood V_K such that $V_K \cap \Gamma = \{1\}$ for every $\Gamma \in \mathcal{L}_K(G)$.*

The conclusion of Theorem 3.2 can equivalently be stated saying that $\mathcal{L}_K(G) \subset \mathrm{Sub}(G)$ is a closed subset of the Chabauty space $\mathrm{Sub}(G)$. The proof consists in studying accumulation points of $\mathcal{L}_K(G)$ in $\mathrm{Sub}(G)$, and can be divided into two independent steps. The first consists in showing that, arguing by contradiction, if $\mathcal{L}_K(G)$ accumulates to a non-discrete subgroup of G, then there is (a finite index subgroup of) one factor G_i that is itself an accumulation point of discrete subgroups in $\mathrm{Sub}(G_i)$ [**CLB19**, Proposition 4.6, Corollary 5.6]. The second step consists in proving the following result, which is of independent interest:

THEOREM 3.3 ([**CLB19**]). *Let G be a non-discrete, compactly generated, QJNC tdlc group. Suppose that there is a sequence of discrete subgroups of G that converges in* $\mathrm{Sub}(G)$ *to a finite index subgroup of G. Then there is a prime p and a compact open subgroup $V \leq G$ such that V is a pro-p group, and V is not topologically finitely generated.*

These two steps indeed give rise to the desired contradiction, because, as already observed in [**BM00a, BM00b**], if a product $G = G_1 \times \cdots \times G_n$ of compactly generated tdlc groups contains a cocompact lattice with a dense projection on each factor, then every compact open subgroup of each G_i must be topologically finitely generated.

The proof of Theorem 3.3 makes use of results of Caprace–Reid–Willis [**CRW17b**], and elaborations of these by Caprace–Reid–Wesolek [**CRW21**], on the local prime contents of compact locally normal subgroups of tdlc groups. Here those results play a role in the abstract setting of Theorem 3.3 that is similar to the role played by Burger-Mozes's generalization of the Thompson-Wielandt theorem in the setting of groups of automorphisms of trees with primitive local action [**BM00a**, Proposition 2.1.2].

Kazhdan–Margulis proved that if G is a semisimple Lie group without compact factor, then there exists an identity neighbourhood V in G such every lattice Γ of G admits a conjugate that intersects V trivially [**KM68**]. This implies in particular the set of covolumes of all lattices in G is bounded away from zero. Here it follows from the conclusion of Theorem 3.2 and Serre's covolume formula (see [**Bou00**, Proposition 1.4.2(b)]) that the set of covolumes $\mathrm{covol}(\Gamma)$, for $\Gamma \in \mathcal{L}_K(G)$, is finite. We do not know whether there could exist a neighbourhood of the identity as in the conclusion of Theorem 3.2 that is actually independent of K.

Under the stronger assumption that each factor is compactly presented (and not only compactly generated), relying on local rigidity results of Gelander-Levit, we obtain a stronger finiteness result than Theorem 3.2, see Theorem C in [**CLB19**].

3. Discrete groups acting on products of graphs

The previous theorems have applications to discrete groups acting on products of graphs. Let X be a connected locally finite graph, and $G \leq \mathrm{Aut}(X)$ be a subgroup of $\mathrm{Aut}(X)$. If $v \in VX$ is a vertex of X, we denote by $X(v)$ the vertices at distance 1 from v, and by G_v the stabilizer of v in G. The permutation group induced by the action of G_v on $X(v)$ is called the **local action** of G at v.

A permutation group L of a set Ω is **primitive** if the only L-invariant partitions of Ω are the trivial ones; **quasi-primitive** if L is transitive and the only intransitive normal subgroup is trivial; and **semi-primitive** if L is transitive and every intransitive normal subgroup of L acts freely. Clearly semi-primitive is a weakening of quasi-primitive, and quasi-primitive is a weakening of primitive.

We say that a subgroup $G \leq \mathrm{Aut}(X)$ has local action with a given property if the local action of G at every vertex has the corresponding property. Note that a subgroup $G \leq \mathrm{Aut}(X)$ with transitive local action always acts cocompactly on X. As first observed by Burger-Mozes, QJNC groups appear naturally in the context of automorphism groups with semi-primitive local action. The following is a simplified version of [**BM00a**, Proposition 1.2.1] (restated in [**CLB19**, Proposition 6.7] for semi-primitive rather than quasi-primitive local action).

PROPOSITION 3.4. *Let X be a connected locally finite graph. Let $G \leq \mathrm{Aut}(X)$ be a closed subgroup with semi-primitive local action. Then G is QJNC.*

Let $G \leq \mathrm{Aut}(X)$ a closed subgroup acting cocompactly on X. If K is a compact subset of G, there is $r \geq 1$ such that any K-cocompact subgroup of G acts on X with at most r orbits. Conversely, for every $r \geq 1$ there is a compact subset $K \subset G$ such that any subgroup of G acting with at most r orbits on X is K-cocompact. Hence in view of Proposition 3.4, the two assertions of the following statement are consequences respectively of Theorem 3.1 and Theorem 3.2.

COROLLARY 3.5 ([**CLB19**]). *Let $n \geq 2$. Let $X_1, \dots, X_n$ be connected locally finite graphs, and for each i let $G_i \leq \mathrm{Aut}(X_i)$ be a non-discrete closed subgroup with semi-primitive local action. Let $G = G_1 \times \cdots \times G_n$ and $X = X_1 \times \cdots \times X_n$.*

(1) For every cocompact lattice $\Gamma \leq G$ such that $p_i(\Gamma)$ is dense in G_i for all i, the set of discrete subgroups of G containing Γ is finite.
(2) For every $r \geq 1$, there exists R such that for every cocompact lattice $\Gamma \leq G$ with at most r orbits on VX and such that $p_i(\Gamma)$ is dense in G_i for all i, we have that the pointwise stabilizer in Γ of any R-ball of X is trivial.

In the setting of discrete groups acting cocompactly on the product of two trees, a version of the Wang finiteness theorem had been previously established by Burger–Mozes in [**BM14**, Theorem 1.1]. Statement 1 extends the main result of [**BM14**] from the case where each G_i has quasi-primitive local action of almost simple type, to the case where each G_i has semi-primitive local action. Burger–Mozes further asked whether, in case each G_i has quasi-primitive local action, the set of covolumes of all cocompact lattices of $G_1 \times G_2$ with dense projections is bounded away from zero [**BM14**, Question 1.2]. Statement 2 provides a partial answer to this question since, $r \geq 1$ being fixed, that statement implies that the set of covolumes of all cocompact lattices of $G_1 \times G_2$ with dense projections and at most r orbits on VX, is finite.

CHAPTER 4

Commensurated subgroups and micro-supported actions

In this chapter we present some results from [**CLB23**], which is joint work with Pierre-Emmanuel Caprace. The reader is invited to consult Appendix A and Appendix B before reading the present chapter.

1. Generalities on commensurated subgroups

Two subgroups Λ, Λ' of a group Γ are **commensurable** if their intersection $\Lambda \cap \Lambda'$ has finite index in Λ and Λ'. A subgroup $\Lambda \leq \Gamma$ is **commensurated** in Γ if all Γ-conjugates of Λ are commensurable. One verifies that this is equivalent to saying that the Λ-action on the coset space Γ/Λ has finite orbits. The most classical example of a commensurated subgroup is $\Lambda = \mathrm{SL}(n, \mathbb{Z})$ inside $\Gamma = \mathrm{SL}(n, \mathbb{Q})$. Every subgroup that is commensurable to a commensurated subgroup is itself commensurated.

Given a group Γ, we are interested in the problem of understanding all (commensurability classes of) commensurated subgroups of Γ. That problem is closely related to the study of homomorphisms $\Gamma \to G$ from Γ to a tdlc group that have dense image (as before tdlc stands for "totally disconnected locally compact"). Indeed, if $\Gamma \to G$ is such a homomorphism, then the preimage in Γ of every compact open subgroup of G is a commensurated subgroup of Γ. Conversely if $\Lambda \leq \Gamma$ is a commensurated subgroup, then one can build a tdlc group G and a homomorphism $\Gamma \to G$ with dense image such that the preimage in Γ of a compact open subgroup of G is commensurable with Λ. One possible way to do this is to consider the left translation action of Γ on the coset space Γ/Λ, and the associated homomorphism $\Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$. Since Λ acts on Γ/Λ with finite orbits, the closure (with respect to pointwise convergence) of the image of Λ in $\mathrm{Sym}(\Gamma/\Lambda)$ is a compact group, and the closure of the image of Γ in $\mathrm{Sym}(\Gamma/\Lambda)$ is tdlc. This tdlc group is denoted $\Gamma /\!\!/ \Lambda$, and is referred to as the Schlichting completion of Γ with respect to Λ [**Sch80**]. We refer to [**SW13**, Section 3] for a more detailed introduction, and to [**RW19**] for additional properties.

Examples of commensurated subgroups are normal subgroups, and more generally subgroups commensurable to a normal subgroup. This includes the finite subgroups, and the finite index subgroups. Subgroups commensurable to a normal subgroup are considered as "trivial examples" of commensurated subgroups. They are characterized by the fact that the completion $\Gamma /\!\!/ \Lambda$ admits a compact open normal subgroup [**CLB23**, Lemma 5.1]. We are generally interested in those commensurated subgroups not of this form.

One important situation where the description of commensurated subgroups remains unknown is when Γ is an irreducible lattice in higher rank semisimple Lie group, or an S-arithmetic subgroup in a product of simple algebraic groups over local fields. While Margulis' normal subgroup theorem says that every normal subgroup of Γ is finite or finite index, the Margulis-Zimmer conjecture on the description of the commensurated subgroups of Γ remains open. Partial positive results were obtained by Venkataramana [**Ven87**] and Shalom-Willis [**SW13**].

Somehow reversing the point of view, if Λ is a discrete subgroup of some ambient locally compact group G, it is of general interest to relate properties of Λ with those of the commensurator $\mathrm{Comm}_G(\Lambda)$ of Λ in G. One key result in this realm is Margulis' arithmeticity theorem, which asserts that if Λ is an irreducible lattice in G a semisimple Lie group with trivial center and no compact factor, then Λ is arithmetic if and only if $\mathrm{Comm}_G(\Lambda)$ is a dense subgroup of G [**Mar91**]. Similar problems relating properties of Λ with those of its commensurator have been investigated in [**LLR11, Mj11, FMvL24**].

2. Commensurated subgroups and the dynamics of micro-supported actions

In [**CLB23**] we consider the problem of the description of the commensurated subgroups in the setting where Γ is a group admitting a faithful, minimal and micro-supported action on a compact space. The main result of [**CLB23**], of which the following statement is a simplified version, relates the existence of certain commensurated subgroups in Γ with the dynamics of the micro-supported actions of Γ.

THEOREM 4.1 ([**CLB23**]). *Let Γ be a finitely generated group such that every proper quotient of Γ is virtually nilpotent. Let X be a compact Γ-space such that the action of Γ on X is faithful, minimal and micro-supported. If Γ has a commensurated subgroup which is of infinite index and which is not virtually contained in a normal subgroup of infinite index of Γ, then the following hold:*

(i) X has a non-empty open subset which is compressible by Γ.
(ii) X is an almost Γ-boundary.

A subgroup $\Lambda \leq \Gamma$ is **virtually contained** in $\Lambda' \leq \Gamma$ if Λ has a finite index subgroup that is contained in Λ'. Let X be a compact Γ-space. Compressibility and Γ-boundaries are defined in Appendix A. The action of Γ on X is an **almost Γ-boundary** if X is minimal and X admits a Γ-invariant clopen partition $X = X_1 \cup \cdots \cup X_d$ such that for each i the action of $\mathrm{Stab}_\Gamma(X_i)$ on X_i is a $\mathrm{Stab}_\Gamma(X_i)$-boundary.

The conclusions of the theorem yield in turn restrictions on the structure of the group Γ. By Lemma B.2, every group admitting a minimal and micro-supported action with a compressible open subset is monolithic, meaning that Γ admits a non-trivial normal subgroup, called the monolith, that is contained in every non-trivial normal subgroup of Γ. For instance Γ cannot be residually finite. Since amenability can be characterized by the property that the only boundary is the one-point space, conclusion (ii) implies that Γ is not amenable.

3. On the proof

The proof of Theorem 4.1 can be divided into several independent steps. In the sequel we sketch the rough outline. From now we take Γ and X as in Theorem 4.1.

Construction of a homomorphism from Γ to a JNC tdlc group. The starting point of the proof is, given a commensurated subgroup Λ of Γ as in the statement, to consider the completion $\Gamma \to \Gamma /\!\!/ \Lambda$, and try to establish a connection between the tdlc group $\Gamma /\!/ \Lambda$ and the Γ-space X. The next step of the proof, described below, associates, given a homomorphism $\Gamma \to G$ from Γ to a tdlc group with dense image, a compact G-space that is built from the compact Γ-space X. It turns out that for this G-space to be non-degenerate and manageable to work with, we will need some assumptions on G. Hence the purpose of this first step of the proof is to replace the homomorphism $\Gamma \to \Gamma /\!/ \Lambda$ we start with by another homomorphism $\Gamma \to G$.

A tdlc group is called just-non-compact (JNC) if it is non compact and every proper quotient is compact. The important feature of compactly generated, non-discrete, JNC tdlc groups is that, by a result of Caprace–Monod, such a group is monolithic, with cocompact monolith [**CM11**] (a result that is not true for discrete groups). For the purpose of the continuation of the proof, the following lemma allows to replace $\Gamma /\!/ \Lambda$ by a compactly generated, non-discrete, JNC group.

Lemma 4.2 ([**CLB23**, Lemma 5.2]). *Let Γ be a finitely generated JNVN group. Let $\Lambda \leq \Gamma$ be a commensurated subgroup which is not virtually contained in a normal subgroup of infinite index of Γ. Then the completion $\Gamma /\!/ \Lambda$ admits a (necessarily compactly generated) non-discrete JNC quotient G, and the composition homomorphism $\varphi\colon \Gamma \to G$ is injective with dense image.*

Construction of a suitable compact G-space. In the sequel we fix $\varphi\colon \Gamma \to G$ an injective homomorphism with dense image from Γ to a second countable tdlc group G. Ideally, one would like to extend the Γ-action on X to a G-action. This is not possible to do so in general. As a substitute, we build a compact G-space such that, viewed as a Γ-space by restricting the action to Γ, will be shown (under certain assumptions) to have properties in common with the Γ-space X.

For, we go through the Chabauty space $\mathrm{Sub}(G)$ of G. We define the map ψ : $X \to \mathrm{Sub}(G)$, $x \mapsto \overline{\varphi(\Gamma_x^0)}$, where Γ_x^0 is the neighbourhood stabilizer of x in Γ (§1.1 in Appendix A). One verifies that ψ is lower semi-continuous and Γ-equivariant [**CLB23**, Lemma 4.1]. Invoking Proposition A.1, we infer that

$$S_{\varphi,G}(X) := \overline{\left\{\overline{\varphi(\Gamma_x^0)} \,:\, x \in X_\varphi\right\}}$$

is a minimal compact Γ-space, where $X_\varphi \subseteq X$ is the set of points where the map $x \mapsto \overline{\varphi(\Gamma_x^0)}$ is continuous. Since Γ is embedded densely in G, $S_{\varphi,G}(X)$ is G-invariant. The issue is that in general $S_{\varphi,G}(X)$ may very well be degenerate, for instance it could a one-point space. This is at this stage that additional assumptions on G come into play, via the following statement, which is a key step of the proof.

Proposition 4.3 ([**CLB23**, Proposition 4.8]). *Assume that G is non-discrete and JNC. Then the action of G on $S_{\varphi,G}(X)$ is faithful and micro-supported.*

Dynamics of micro-supported actions of JNC tdlc groups. A subgroup of a tdlc group G whose normalizer is open is called locally normal. Two closed subgroups K, L of G are called locally equivalent if $K \cap L$ is relatively open in both K and L. Following [**CRW17a**], the structure lattice $\mathcal{LN}(G)$ is defined as the set of local classes of closed locally normal subgroups of G. The group G acts by conjugation on $\mathcal{LN}(G)$. The centralizer lattice of G, denoted $\mathcal{LC}(G)$, is the subset of $\mathcal{LN}(G)$ consisting of the local classes of centralizers of locally normal subgroups of G. It is shown in [**CRW17a**] that, under certain assumptions on G (which are satisfied for instance if G is compactly generated, non-discrete and JNC), then the map $^{\perp}$:

$\mathcal{LC}(G) \to \mathcal{LC}(G)$, $[K]^\perp = [C_G(K)]$, is well-defined and the operations

$$[K] \wedge [L] = [K \cap L]$$

and

$$[K] \vee [L] = \left([K]^\perp \wedge [L]^\perp\right)^\perp$$

turn $\mathcal{LC}(G)$ into a Boolean algebra. The Stone space of the Boolean algebra $\mathcal{LC}(G)$ is denoted by Ω_G. It is a compact totally disconnected G-space. The dynamics of the action of G on Ω_G has been studied in [**CRW17b**] under the assumption that the G-action on Ω_G is faithful. The following strongly relies on [**CRW17b**] and [**CM11**].

THEOREM 4.4 ([**CLB23**, Theorem 3.17]). *Let G be a compactly generated monolithic tdlc group, and assume that the monolith of G is compactly generated, non-compact and non-discrete (e.g. G is non-discrete JNC).*

(i) If $\mathcal{LC}(G)$ is infinite, then the G-action on Ω_G is faithful.

Moreover, any totally disconnected compact G-space X on which the G-action is faithful and micro-supported is a factor of Ω_G, and enjoys the following properties:

(ii) The G-action on X has a compressible clopen subset.

(iii) The G-action on X is an almost G-boundary.

Conclusion. Coming back to our sketch of proof of Theorem 4.1, faithfulness of the G-action on $S_{\varphi,G}(X)$ given by Proposition 4.3 ensures $\mathcal{LC}(G)$ is infinite, and hence the conclusion of Theorem 4.4 holds for the G-action on $S_{\varphi,G}(X)$. The subgroup Γ being dense in G, we infer that the Γ-action on $S_{\varphi,G}(X)$ also is an almost boundary with compressible open subsets. To relate this action to the original action of Γ on X, we invoke a theorem of Rubin [**Rub96**], which ensures that the Γ-spaces X and $S_{\varphi,G}(X)$ admit a common highly proximal extension. We finally conclude via the fact that the property of being an almost boundary with compressible open subset is inherited by (and from) a highly proximal extension [**CLB23**, §2.2].

4. Applications

In this subsection we describe concrete situations where Theorem 4.1 or other results from [**CLB23**] apply.

Topological full groups. Let Λ be a group acting on a Cantor space X. The associated topological full group is denoted $\mathbb{F}(\Lambda, X)$. We also denote by $\mathbb{A}(\Lambda, X) \leq \mathbb{F}(\Lambda, X)$ the alternating full group introduced and studied by Nekrashevych in [**Nek19, Nek18**]. The action of Λ on X expansive if there exist a compatible metric d and $\delta > 0$ such that for every $x \neq y \in X$ there exists $\gamma \in \Lambda$ such that $d(\gamma(x), \gamma(y)) \geq \delta$. If Λ is a finitely generated group and $\Lambda \curvearrowright X$ is a minimal and expansive, Nekrashevych showed that the group $\mathbb{A}(\Lambda, X)$ is the monolith of $\mathbb{F}(\Lambda, X)$, and is a finitely generated and simple group. When $\Lambda = \mathbb{Z}$ the alternating full group coincides with the derived subgroup of $\mathbb{F}(\Lambda, X)$, and in that case finite generation and simplicity of $\mathbb{F}(\Lambda, X)'$ were previously obtained by Matui in [**Mat06**].

Suppose in addition that X admits a probability measure that is invariant under the action of Λ. Such a probability measure is necessarily also invariant under the action $\mathbb{F}(\Lambda, X)$. Since the existence of an invariant probability measure is incompatible with the conclusion of Theorem 4.1, we obtain:

Corollary 4.5 ([**CLB23**]). *Let Λ be a finitely generated group, and $\Lambda \curvearrowright X$ be a minimal and expansive action on a Cantor space X. If X carries a Λ-invariant probability measure (e.g. if Λ is amenable), then every proper commensurated subgroup of the alternating full group $\mathbb{A}(\Lambda, X)$ is finite.*

Corollary 4.5 implies that for every minimal and expansive action of $\mathbb{Z}^d$ on a Cantor space, every proper commensurated subgroup of $\mathbb{A}(\mathbb{Z}^d, X)$ is finite. When $d = 1$, Juschenko and Monod showed that the group $\mathbb{F}(\mathbb{Z}, X)$ is amenable [**JM13**]. One motivation for studying specifically the commensurated subgroups of finitely generated infinite simple amenable groups comes from the fact that, if such a group Γ were known to admit an infinite proper commensurated subgroup Λ, then by results of [**CM11**], the Schlichting completion $\Gamma /\!/ \Lambda$ would admit as a quotient a compactly generated tdlc group that is non-discrete, topologically simple and amenable (see Proposition 3.6 in [**CRW17b**]). Equivalently, using notation from Section 5 of Chapter 1, a group that belongs to the class $\mathcal{S}$ and is amenable. As of now, no such example is available (see Question 3 in [**CRW17b**]). Hence Corollary 4.5 implies that the above strategy to build such a group cannot work by starting with groups such as $\mathbb{A}(\Lambda, X)$. Corollary 4.5 also answers a question raised at the end of [**Cap18**].

Groups acting on the circle. We refer the reader to [**Ghy01**] for an introduction to group actions on the circle.

Theorem 4.6 ([**CLB23**]). *Let Γ be a finitely generated group with a minimal and micro-supported action on the circle. Suppose that the subgroup generated by the elements $\gamma \in \Gamma$ such that there is a non-empty open interval that is fixed by γ has finite index in Γ. Then every commensurated subgroup of Γ is commensurable to a normal subgroup of Γ.*

An example of group to which the result applies is Thompson's group T, yielding that every proper commensurated subgroup of Thompson's group T is finite. In that case this result had been previously established in joint work with Wesolek [**LBW19**].

Branch groups. We refer the reader to [**Gri00, BGS03**] for a general introduction to branch groups. The proof of the following statement relies on the main result from [**CLB23**], as well as the fact that every normal subgroup of a finitely generated branch group is finitely generated: that property is established by Francoeur in the appendix of [**CLB23**].

Theorem 4.7 ([**CLB23**]). *Let Γ be a finitely generated branch group. Then every commensurated subgroup of Γ is commensurable to a normal subgroup.*

Theorem 4.7 recovers and extends a result of Wesolek [**Wes17**], who showed the same result under the additional assumption that Γ is just-infinite.

Micro-supported actions of locally compact groups. The results from [**CLB23**] also have applications to the study of non-discrete locally compact groups. Let G be a compactly generated tdlc group that is non-discrete and topologically simple (retaining previous notation, G is a group in the class $\mathcal{S}$). The centralizer lattice $\mathcal{LC}(G)$ of G is well-defined [**CRW17a**]. The first question one wants to address is whether the Stone space Ω_G is trivial (i.e. a one-point space).

Now suppose G, H are two groups as above, and there exists a continuous homomorphism $\varphi\colon H \to G$ with dense image. Examples of such pairs (H, G) naturally arise in the study of the local structure of groups in the class $\mathcal{S}$, see [**Rei13**, Theorem 1.2] and [**CRW21**]. In this situation, we study the relationship between the centralizer lattice of H and that of G. We show:

THEOREM 4.8 ([**CLB23**]). *Let G, H be compactly generated tdlc groups that are non-discrete and topologically simple, and $\varphi\colon H \to G$ be a continuous injective homomorphism with dense image. Suppose Ω_H is non-trivial. Then Ω_G is non-trivial, and the H-action on Ω_G is micro-supported.*

5. Other works (directly or indirectly) related to Chapter 4

(1) On commensurators of free groups and free pro-p groups (with Y. Barnea, M. Ershov, C. Reid, M. Vannacci, T. Weigel). https://arxiv.org/abs/2507.04120.
(2) Commensurated subgroups in tree almost automorphism groups (with Ph. Wesolek).
Groups Geom. Dyn.,Vol. 13, Issue 1 (2019), 1-30.

CHAPTER 5

Rigidity and flexibility results for groups with a common envelope

In this chapter we present results from [**LB25**]. The reader is invited to consult Appendix C before reading the present chapter.

1. Terminology

A locally compact group G is a **cocompact envelope** of a discrete group Γ if G contains a discrete and cocompact subgroup isomorphic to Γ [**Fur67**]. We will say that two discrete groups Γ and Λ share a cocompact envelope if there is a locally compact group G such that G is a common cocompact envelope of Γ and Λ.

Definition 5.1. Let $\mathcal{C}$ be a class of finitely generated groups. We say that $\mathcal{C}$ is rigid for cocompact envelopes if for every group Λ that shares a cocompact envelope with a group Γ in $\mathcal{C}$, the group Λ is virtually isomorphic up to finite kernel to a group in $\mathcal{C}$. We write "CE-rigid" for "rigid for cocompact envelopes".

A class $\mathcal{C}$ is QI-rigid if for every group Λ that is QI to a group Γ in $\mathcal{C}$, the group Λ is virtually isomorphic up to finite kernel to a group in $\mathcal{C}$. Two groups sharing a cocompact envelope are always QI, so if the class $\mathcal{C}$ is QI-rigid, then $\mathcal{C}$ is CE-rigid. Hence the question of CE-rigidity of a given class is relevant either when QI-rigidity is not known, or when QI-rigidity is known to fail. The problem of CE-rigidity is a sub-problem of the more general problem of studying all finitely generated groups, or more generally compactly generated locally compact groups, that are commable to a group Γ in $\mathcal{C}$.

2. Nilpotent normal subgroup

In this section we consider CE-rigidity in the situation where the common cocompact envelope G of two groups Γ and Λ is a totally disconnected locally compact (tdlc) group. And we deal with the class of groups Γ having a finitely generated nilpotent normal subgroup (implicitly assumed to be infinite, otherwise that requirement is void).

Bader–Furman–Sauer showed that, under certain group theoretic requirements on a group Γ, the envelopes G of Γ in which the connected component of the identity G^0 is not compact, enjoy very strong restrictions. See [**BFS20**, Theorem A]. This result reduces, for those groups Γ, the general study of envelopes of Γ to the study of tdlc envelopes. We also note that among those aforementioned group theoretic requirements on Γ for [**BFS20**, Theorem A] to hold, there is the fact that Γ does not admit any infinite amenable commensurated subgroup. Hence the setting of the present section, where the ambient group G is tdlc and the subgroup Γ does admit an infinite normal amenable (indeed nilpotent) subgroup, is disjoint from (and hence complementary to) the setting of [**BFS20**, Theorem A].

For finitely generated groups Γ and Λ, the existence of a common tdlc cocompact envelope is equivalent to the existence of a connected locally finite graph on which Γ and Λ act faithfully and geometrically. Indeed, if X is such a graph, then the group of automorphisms of X is a common totally disconnected cocompact envelope. And conversely if G is such an envelope, then Γ and Λ act geometrically on any Cayley-Abels graph $\mathrm{CayAb}(G, U, S)$ of G, and Γ and Λ act faithfully on $\mathrm{CayAb}(G, U, S)$ provided the compact open subgroup U of G is sufficiently small.

THEOREM 5.2 ([**LB25**]). *Let Γ be a finitely generated group with a normal subgroup $A \lhd \Gamma$ such that A is finitely generated and nilpotent. Suppose that Γ and Λ share a tdlc cocompact envelope. Then there is a finite index subgroup Λ' of Λ such that Λ' admits a normal subgroup $B \lhd \Lambda'$ such that A and B are virtually isomorphic.*

Given the above reformulation in terms of geometric actions on graphs, Theorem 5.2 says that the property of containing a finitely generated nilpotent normal subgroup (of a given virtual isomorphism class) can be detected on the graphs on which the group acts geometrically (modulo passing to finite index).

It is worth comparing Theorem 5.2 with the situation where the common cocompact envelope is not necessarily totally disconnected. The groups studied by Leary-Minasyan in [**LM21**] include examples of groups Γ and Λ acting faithfully and geometrically on $\mathbb{R}^d \times T$ - the product of the d-dimensional Euclidean space $\mathbb{R}^d$ and a locally finite tree T - such that Γ is of the form $\Gamma = A \times F$ with $A = \mathbb{Z}^d$ and F is a non-abelian free group of finite rank, and such that Λ does not virtually admit any non-trivial abelian (or nilpotent) normal subgroup. Here the common cocompact envelope is $\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Aut}(T)$. Hence these examples show that Theorem 5.2 does not hold when the common envelope is not totally disconnected.

It is also interesting to connect the setting of Theorem 5.2 with other rigidity results from the literature. Mosher–Sageev-Whyte's Theorem 2 in [**MSW03**] shows that if a group Γ acts cocompactly on an infinitely ended locally finite tree such that vertex stabilizers (which are commensurated subgroups of Γ) are finitely generated and nilpotent, and if Λ is a group QI to Γ, then Λ acts cocompactly on an infinitely ended locally finite tree with vertex stabilizers QI to those of Γ. That result has been generalized by Margolis [**Mar21**]. The setting of [**Mar21**, Theorem 1.4] covers the situation of a group Γ having a finitely generated nilpotent subgroup A that is commensurated in Γ, and provides sufficient conditions under which every group Λ that is QI to Γ admits a finitely generated nilpotent subgroup B such that B is commensurated in Λ and B is QI to A. So [**MSW03**, Theorem 2] and [**Mar21**, Theorem 1.4] both consider commensurated subgroups, while Theorem 5.2 deals with normal subgroups (both in the assumption and in the conclusion). These two results hold in the more general setting of QI groups Γ, Λ, as opposed to the stronger assumption in Theorem 5.2 that Γ, Λ share a cocompact tdlc envelope. [**Mar21**, Theorem 1.4] requires on the one hand Γ to be of type F_n for a certain $n \geq 2$ (depending on A), and on the other hand that the coset space Γ/A has infinitely many ends (the case where Γ/A is a tree corresponding to [**MSW03**, Theorem 2]). Theorem 5.2 has no such assumption.

A main tool in the proof of Theorem 5.2, which also plays a major role in the proof of Theorem 5.7 below, is provided by the following result of independent interest.

THEOREM 5.3 ([**LB25**]). *Let G be a tdlc group, and Γ a (not necessarily closed) subgroup of G such that there is a compact subset K of G such that $G = \Gamma K$. If A is a finitely generated nilpotent subgroup of G that is normalized by Γ, then there exists a compact open subgroup U of G such that U is normalized by A.*

The proof of Theorem 5.3 makes use of Willis' theory [**Wil94, Wil01, Wil04**]. From that perspective, Theorem 5.3 can be thought of as a poorness property of the dynamics of the global conjugation action of A on G, as it means that there is an invariant compact neighbourhood of the identity. The original form of Willis' work deals with the study of the conjugation action of an individual element g on a tdlc group G. The important notion there is the notion of tidy subgroups. We provide a brief discussion in §2.1 in [**LB25**] about their relevant properties. When we move in the study of the conjugation action on G from the case of an individual acting element g to the case of a subgroup A acting on G, in general very few tools are available. However when A is finitely generated nilpotent or polycyclic, results of Shalom–Willis ensure the existence of tidy subgroups common for all elements of A [**SW13**]. The proof of Theorem 5.3 notably relies on these results.

3. Solvable groups of finite rank

As observed by Erschler, the class of finitely generated solvable groups is not CE-rigid. If F_1, F_2 are two finite groups of the same cardinality, the wreath products $\Gamma = F_1 \wr \mathbb{Z}$ and $\Lambda = F_2 \wr \mathbb{Z}$ admit a common Cayley graph, and hence a common cocompact envelope (the automorphism group of this Cayley graph). And Γ is solvable provided F_1 is, while Λ is not virtually solvable provided F_2 is not solvable [**Dyu00**].

Here we focus on the class of solvable groups of finite rank (finite Prüfer rank). We refer to Section 3 of Chapter 2 for a brief discussion on solvable groups of finite rank, and to [**LR04**] for more general background. The large-scale geometry and QI-rigidity of certain families of non-polycyclic solvable groups of finite rank have been studied by Farb–Mosher in [**FM98, FM99, FM00a**]. In the case of the solvable Baumslag–Solitar group $\Gamma = \mathbb{Z}[1/n] \rtimes_n \mathbb{Z}$, $n \geq 2$, the main result of [**FM99**] asserts that if Λ is a group QI to Γ, then Λ is virtually isomorphic up to finite kernel to Γ.

In [**LB25**] we establish both positive and negative results regarding CE-rigidity for the class of solvable groups of finite rank. In the direction of rigidity, we have the following:

THEOREM 5.4 ([**LB25**]). *Let Γ be a finitely generated solvable group of finite rank. Let Λ such that Γ and Λ share a cocompact envelope. Suppose that Λ has no normal subgroup that is infinite and locally finite. Then Λ is virtually solvable of finite rank.*

On the flexibility side, we provide in [**LB25**] two constructions that show CE-rigidity fails for the class of solvable groups of finite rank. So the assumption in Theorem 5.4 that Λ has no normal subgroup that is infinite and locally finite is truly needed. Our first construction yields examples of finitely generated groups Γ, Λ that share a cocompact envelope, with Γ solvable of finite rank, and Λ not virtually solvable [**LB25**, Theorem 5.10]. We exhibit examples with Γ of the form $\Gamma = \mathbb{Z}[1/n]^2 \rtimes \mathbb{Z}$ for $n \geq 2$ (so Γ is abelian-by-$\mathbb{Z}$ and of Hirsch number 3), and Λ of the form $\mathbb{Z}^2 \times F \wr \mathbb{Z}$, where F is an arbitrary finite group of cardinality n.

The above examples of groups Γ and Λ are not finitely presented. However we provide in [**LB25**] a second construction that shows CE-rigidity fails for the class of solvable groups of finite rank, that includes finitely presented groups, and even groups with higher finiteness properties. Recall that a group Γ has type F_n, $n \geq 1$, if there exists a CW-complex with finite n-skeleton, with fundamental group Γ and contractible universal cover [**Wal65**]. F_{n+1} implies F_n, and F_1 and F_2 are respectively equivalent to being finitely generated and being finitely presented. Having type F_n is a QI-invariant [**DK18**, Theorem 9.56].

Theorem 5.5 ([**LB25**]). *For every $n \geq 1$, there are groups Γ, Λ of type F_n such that Γ, Λ share a cocompact envelope, and:*

- Γ, Λ *are solvable;*
- Γ *is of finite rank and torsion-free;*
- Λ *is neither of finite rank nor virtually torsion-free.*

As a consequence, we deduce:

Corollary 5.6 ([**LB25**]). *The class of finitely generated solvable groups of finite rank is not QI-rigid. More generally, for every $n \geq 1$, the class of solvable groups of finite rank of type F_n is not QI-rigid.*

As a consequence of Gromov's polynomial growth theorem, the class of finitely generated nilpotent groups is QI-rigid. It is an open question whether the class of polycyclic groups is QI-rigid [**FM00b, Sha04, EFW07, EF10**]. It is therefore natural to ask about QI-rigidity for classes of solvable groups as close as possible to the class of polycyclic groups. The above corollary establishes a border to what we can hope for. Eskin–Fisher–Whyte conjectured that the class of polycyclic groups shall be QI-rigid [**EFW07, EF10**, Conjecture 1.2]. Positive results in this direction had been obtained by Shalom, who showed that any infinite group QI to a polycyclic group has a positive first virtual Betti number [**Sha04**]. Eskin–Fisher–Whyte showed that the subclass of the class of polycyclic groups consisting of cocompact lattices in the Lie group Sol, is QI-rigid [**EFW12, EFW13**]. Generalizations to higher dimensional examples have been obtained by Peng [**Pen11a, Pen11b**].

Farb–Mosher showed that the class of non-polycyclic finitely presented groups that are abelian-by-$\mathbb{Z}$ (such groups are necessarily of finite rank) is QI-rigid [**FM00a**]. The fact that the class finitely presented solvable groups of finite rank is not QI-rigid (i.e. Corollary 5.6 for $n = 2$) implies that this QI-rigidity result no longer holds beyond the abelian-by-$\mathbb{Z}$ case.

Shalom showed that if Γ is solvable of finite rank, and if Λ is a group QI to Γ such that Λ is solvable and torsion-free, then Λ is of finite rank [**Sha04**, Theorem 1.6]. Theorem 5.5 shows that this theorem no longer holds without the assumption that Λ is torsion-free.

The phenomena exhibited in Theorem 5.5 also show that the class of finitely presented torsion-free solvable groups, and more generally the class of torsion-free solvable groups of type F_n, is not QI-rigid [**LB25**, Corollary 6].

Our flexibility results that lead to Theorem 5.5 consist in embedding some solvable groups of finite rank Γ as cocompact lattices in some (necessarily amenable) locally compact group G, where G has the structure of a product $G = G_{\mathrm{c}} \times G_{\mathrm{td}}$. The factor G_{c} is a virtually connected solvable Lie group, and the factor G_{td} is a tdlc group.

The group Γ is embedded as an irreducible lattice in G. On the other hand, things are arranged so that each one of the factors G_{c} and G_{td} admits a cocompact lattice such that the product of these produces a group Λ that fails to be virtually solvable of finite rank.

Our constructions involve the Diestel–Leader graphs $\mathrm{DL}_d(n)$, i.e. the subset of the product of d copies of an $(n+1)$-regular tree $T_1 \times \cdots \times T_d$ defined by the equation $b_1 + \cdots + b_d = 0$, where b_i is a Busemann function on T_i. We provide a first construction where $G_{\mathrm{c}} = \mathrm{Isom}(\mathbb{R}^2)$, the group of isometries of $\mathbb{R}^2$, and $G_{\mathrm{td}} = \mathrm{Isom}(\mathrm{DL}_2(n))$. This construction provides the examples mentioned in the paragraph right after Theorem 5.4. We provide another construction (that leads to Theorem 5.5) where $G_{\mathrm{c}} = \mathbb{R}^d \rtimes (\mathbb{R}^\times)^{d-1}$, where $(\mathbb{R}^\times)^{d-1}$ is identified with the group of $(d \times d)$-diagonal matrices of determinant one; and where $G_{\mathrm{td}} = \mathrm{Isom}(\mathrm{DL}_d(n_1)) \times \cdots \times \mathrm{Isom}(\mathrm{DL}_d(n_k))$ is a product of isometry groups of Diestel–Leader graphs. And $d \geq 2$ and $k \geq 1$ are arbitrary. As a concrete example, our smallest finitely presented groups Γ, Λ as in Theorem 5.5 are obtained with $d = 3$ and $k = 1$ and are of the form $\Gamma = \mathbb{Z}[1/p]^3 \rtimes \mathbb{Z}^4$ and $\Lambda = \mathbb{Z}^3 \rtimes \mathbb{Z}^2 \times \mathbb{F}_p[t, t^{-1}, (t+1)^{-1}] \rtimes \mathbb{Z}^2$. See Examples 5.15 in [**LB25**] for a detailed example. In Λ the right factor $\mathbb{F}_p[t, t^{-1}, (t+1)^{-1}] \rtimes \mathbb{Z}^2$ is Baumslag's metabelian group from [**Bau72**].

4. Polycyclic groups

We now turn our attention to polycyclic groups.

THEOREM 5.7 ([**LB25**]). *Consider the class of locally compact groups G such that, after modding out by a compact normal subgroup, we have:*

(1) G is unimodular and amenable;

(2) the identity component G^0 is open in G, and G/G^0 is virtually polycyclic.

Then this class is stable under commability.

One consequence of Theorem 5.7 is that if Γ is polycyclic and G is a cocompact envelope of Γ, then after modding out by a compact normal subgroup, G is an extension of a connected Lie group and a discrete group. Hence another special case of Theorem 5.7 (which is actually an intermediate step in the proof) is that every tdlc cocompact envelope of Γ is compact-by-discrete. Recall that a group is compact-by-discrete if it has a compact normal subgroup whose associated quotient is discrete. Envelopes that are compact-by-discrete are somehow the trivial envelopes.

A discrete group belongs to the class described in the theorem if and only if it is virtually polycyclic. Hence:

COROLLARY 5.8 ([**LB25**]). *Let Γ be a polycyclic group, and let Λ be a discrete group that is commable to Γ. Then Λ is virtually polycyclic.*

For certain polycyclic groups Γ, we show that the only cocompact envelopes are the ones that live within a certain natural envelope. Let $d \geq 2$, and let $A \leq \mathrm{SL}(d, \mathbb{Z})$ be a subgroup such that $A \simeq \mathbb{Z}^k$ and every non-trivial element of A has d distinct positive eigenvalues. So A is diagonalizable over $\mathbb{R}$, and there are commuting real matrices $X_1, \ldots, X_k$ such that, if $\varphi : \mathbb{R}^k \to \mathrm{GL}(d, \mathbb{R})$ is the homomorphism defined by $\varphi(t_1, \ldots, t_k) = \exp(t_1 X_1 + \cdots + t_k X_k)$, then $\varphi(\mathbb{Z}^k) = A$. The associated connected Lie group $G_\varphi = \mathbb{R}^d \rtimes \mathbb{R}^k$ is then a cocompact envelope of the group $\Gamma = \mathbb{Z}^d \rtimes A \simeq \mathbb{Z}^d \rtimes \mathbb{Z}^k$. We show the following:

THEOREM 5.9 ([**LB25**]). *Suppose that every non-trivial element of A has d distinct eigenvalues, which are positive and different from 1. Then every cocompact envelope G of $\Gamma = \mathbb{Z}^d \rtimes A$ is, up to passing to a finite index subgroup and modding out by a compact normal subgroup, isomorphic to a closed cocompact subgroup of $G_\varphi = \mathbb{R}^d \rtimes \mathbb{R}^k$.*

This theorem generalizes a result of Dymarz, who showed Theorem 5.9 for $k = 1$. Specifically, when $d = 2$ and $k = 1$, the group $G_\varphi = \mathbb{R}^2 \rtimes \mathbb{R}$ is the Lie group Sol, and in that case Theorem 5.9 is [**Dym15**, Theorem 1.1.1]. More generally, the case $d \geq 2$ and $k = 1$ in Theorem 5.9 is covered by [**Dym15**, Theorem 1.2].

CHAPTER 6

Commability of Baumslag-Solitar groups and generalizations

In this chapter we present results from [**CLB**], which is joint work with Yves Cornulier. The reader is invited to consult Appendix C before reading the present chapter.

1. The setting

For a fixed $n \geq 1$, consider the class $\mathcal{D}_n$ of finitely generated groups admitting a cocompact action on an infinitely ended locally finite tree such that vertex stabilizers are virtually $\mathbb{Z}^n$. In the terminology of [**MSW03**], these are homogeneous graphs of groups with vertex groups virtually $\mathbb{Z}^n$. These groups are notably studied in [**Kro90, For03, Lev07**]. For $n = 1$, examples of groups in $\mathcal{D}_1$ are Baumslag–Solitar groups $\mathrm{BS}(p, q) = \langle t, x \mid tx^p t^{-1} = x^q \rangle$, where $0 < |p| < |q|$.

The behaviours of groups in the class $\mathcal{D}_n$ with respect to quasi-isometries has been intensively studied. Mosher–Sageev–Whyte showed that every finitely generated group that is QI to a group in $\mathcal{D}_n$ must belong to $\mathcal{D}_n$ [**MSW03**]. As for the internal QI classification, the situation splits into two distinct behaviours. In the virtually solvable case, Farb–Mosher showed that the situation is very rigid [**FM99**]. In the non-virtually solvable case, Whyte showed that many groups in $\mathcal{D}_n$ are QI to each other, and obtained a description of QI classes [**Why01, Why10**]. The precise statement of Whyte's result is discussed later.

The main purpose of [**CLB**] is to address the classification of groups in $\mathcal{D}_n$ up to commability, with the wish to compare it with Whyte's classification up to QI. For, we exhibit an appropriate class of locally compact groups, that is shown to be invariant under commability, and in which we are able to extract a commability invariant.

Definition 6.1. For $n \geq 1$, let I_n be the class of locally compact groups O such that O has a maximal compact normal subgroup K_O, and O/K_O is virtually isomorphic to $\mathbb{Z}^p \times \mathbb{R}^q$ with $p + q = n$.

When $n = 1$, O/K_O is virtually isomorphic either to $\mathbb{Z}$ or to $\mathbb{R}$, and the class I_1 is precisely the class of locally compact groups with two ends. It also coincides with the class of locally compact groups QI to $\mathbb{Z}$. However when $n \geq 2$, the class I_n is a proper subclass of the class of locally compact groups QI to $\mathbb{Z}^n$ (for instance the group $\mathrm{Isom}(\mathbb{R}^n)$ is not in I_n).

Definition 6.2. For $n \geq 1$, let $T(n)$ be the class of locally compact groups G such that G admits a continuous cocompact action on an infinitely ended locally finite tree, such that vertex stabilizers are in I_n.

We note that a discrete group is in I_n if and only if it is virtually $\mathbb{Z}^n$. Hence the discrete groups in $T(n)$ are precisely the groups in the class $\mathcal{D}_n$ mentioned above.

2. The one-dimensional case

We first discuss the case $n = 1$. If $O \in I_1$ and K_O the maximal compact normal subgroup, one verifies that the homomorphism $O \to O/K_O$ induces a homomorphism at the level of commensurators $\mathrm{Comm}(O) \to \mathrm{Comm}(O/K_O)$. The latter group is either $\mathbb{Q}^\times$ or $\mathbb{R}^\times$; according to whether O/K_O is virtually $\mathbb{Z}$ or $\mathbb{R}$. In any case, viewing $\mathbb{Q}^\times$ as a subgroup of $\mathbb{R}^\times$, we have a homomorphism $\mathrm{Comm}(O) \to \mathbb{R}^\times$.

Let $G \in T(1)$. Take T a tree on which G acts as in the definition, and G_v a vertex stabilizer. Since the tree T is locally finite, vertex stabilizers in G are all commensurable with each other. In particular these are commensurated subgroups of G. Hence we have a homomorphism $G \to \mathrm{Comm}(G_v)$. Composed with $\mathrm{Comm}(G_v) \to \mathbb{R}^\times$, we obtain a homomorphism $\rho_G : G \to \mathbb{R}^\times$. One verifies that $\rho_G : G \to \mathbb{R}^\times$ does not depend on the choice of T. For the group $\Gamma = \mathrm{BS}(p, q)$, the image of ρ_Γ is the cyclic subgroup of $\mathbb{R}^\times$ generated by p/q.

Theorem 6.3 ([**CLB**]). *Let $\mathcal{C}_1$ denote the class of groups G in $T(1)$ such that ρ_G has infinite image. Then the class $\mathcal{C}_1$ is stable under commability, and if $G, H \in \mathcal{C}_1$ are commable, then the subgroups $\rho_G(G)$ and $\rho_H(H)$ of $\mathbb{R}^\times$ are commensurable.*

After the preprint [**CLB**] started circulating, Tessera informed us that Carette–Tessera had previously proven Theorem 6.3, as well as the converse of that statement, several years ago (unpublished).

Note that it is clear from the definition that the class $T(1)$ is stable under taking quotient by a compact normal subgroup and forming an extension by a compact normal subgroup. Also it is easy to see that if G belongs to $T(1)$ and H is a closed cocompact subgroup of G, then H belongs to $T(1)$ (just by restricting the defining G-action). The contents of the theorem is that if H belongs to $T(1)$ (more precisely, $\mathcal{C}_1$), then so does G (and we have the conclusion about $\rho_G(G)$ and $\rho_H(H)$).

3. The case $n \geq 2$

The general case is more involved than the one-dimensional case. If G is a group in $T(n)$, then similarly as in the case $n = 1$ one can define a homomorphism $\rho_G : G \to \mathrm{GL}(n, \mathbb{R})$ (where we have identified $\mathrm{Comm}(O/K_O)$ with a subgroup of $\mathrm{GL}(n, \mathbb{R})$, which admits as a consequence that this homomorphism is well-defined up to conjugation in $\mathrm{GL}(n, \mathbb{R})$). This extends the previous definition for $n = 1$. For discrete groups in $T(n)$, this homomorphism was defined by Whyte and plays a key role in [**Why10**].

We are lead to consider the following properties. Given a representation $\rho : G \to \mathrm{GL}(n, \mathbb{R})$, consider the conditions:

- (I) The only d such that there exists a representation $\sigma : G \to \mathrm{GL}(d, \mathbb{R})$ with the property that $\sigma(G)$ is relatively compact in $\mathrm{GL}(d, \mathbb{R})$ and there is a linear sujrective G-equivariant map $\mathbb{R}^n \to \mathbb{R}^d$; is $d = 0$.
- (II) no finite index subgroup of $\rho(G)$ normalizes a non-trivial compact connected subgroup of $\mathrm{GL}(n, \mathbb{R})$.

A sufficient condition implying (I) and (II) is that the representation has Zariski-dense image.

Theorem 6.4 ([**CLB**]). *For $n \geq 1$, let $\mathcal{C}_n$ denote the class of groups G in $T(n)$ such that $\rho_G : G \to \mathrm{GL}(n, \mathbb{R})$ verifies (I) and (II). Then the class $\mathcal{C}_n$ is stable under* *commability, and if $G, H \in \mathcal{C}_n$ are commable then the images of ρ_H and ρ_G are commensurable up to conjugation in* $\mathrm{GL}(n, \mathbb{R})$.

We make some comments:

- For $n = 1$, condition (I) is equivalent to asking that the representation has infinite image, and (II) is void. Hence the definition of $\mathcal{C}_1$ in the previous statement is consistent with the one given in Theorem 6.3, and Theorem 6.4 for $n = 1$ reduces to Theorem 6.3.
- (Leary-Minasyan examples [**LM21**], already mentioned in Section 2 of Chapter 5). For each $n \geq 2$, there exists a finitely generated group Γ that acts geometrically on $\mathbb{R}^n \times T$ (where T is a locally finite regular tree of degree ≥ 3), such that the Γ-action on each factor $\mathbb{R}^n$ and T is not proper. Such a group belongs to $T(n)$, and the representation $\rho_\Gamma : \Gamma \to \mathrm{GL}(n, \mathbb{R})$ has infinite image (contained in a compact subgroup). The group Γ is therefore commable to $\Lambda = \mathbb{Z}^n \times F_k$, where F_k is a free discrete cocompact subgroup in $\mathrm{Aut}(T)$. The group Λ is also in $T(n)$, and the representation $\rho_\Lambda : \Lambda \to \mathrm{GL}(n, \mathbb{R})$ is trivial. So these examples show that Theorem 6.4 does not hold without any assumption on the representation ρ_G.

Comparison with the QI classification. As mentioned above, Whyte has classified the non-virtually solvable discrete groups in $T(n)$ up to QI. When $n = 1$, the result is that all non-virtually solvable discrete groups in $T(1)$ that are not virtually $\mathbb{Z} \times F_k$ are QI to each other [**Why01**]. In higher dimension, Whyte showed that if Γ, Λ in $T(n)$ are QI, then the images of ρ_Γ and ρ_Λ are, up to conjugation, Hausdorff equivalent (i.e. at bounded Hausdorff distance). Moreover the Hausdorff equivalence class almost determines the QI class of Γ [**Why10**].

It is interesting to compare Whyte's classification with Theorem 6.4, because the latter says that the commability class of a group Γ as in the theorem detects a finer invariant than the Hausdorff equivalence class of the image of ρ_Γ, namely the commensurability class of the image of ρ_Γ. So for instance the groups $\mathrm{BS}(p, q)$, $0 < |p| < |q|$, are all QI to each other, but fall into infinitely many commability classes. Similarly one obtains for every $n \geq 1$ infinitely many groups in $T(n)$ that are all QI to each other but pairwise not commable.

The question of finding examples of finitely generated groups that are QI but not commable had been originally asked by Cornulier. Examples have been constructed by Carette and Tessera [**Cor18**, 19.5.2]. They showed if G is a connected semisimple Lie group with trivial center and no compact factor and Γ_1, Λ_1 are two discrete and cocompact subgroups of G that are not virtually isomorphic, then the groups $\Gamma = \Gamma_1 * \mathbb{Z}$ and $\Lambda = \Lambda_1 * \mathbb{Z}$ are QI but not commable. Their argument for showing that Γ and Λ are not commable makes crucial use of the fact that these groups have infinitely many ends. The above examples are of quite different nature. In particular they are one-ended.

Lattice embeddings. Our results also provide information, given a discrete group Γ in $T(n)$, on the locally compact groups into which Γ can embed as a lattice. We show:

THEOREM 6.5 ([**CLB**]). *Let $m, n \geq 1$. Let Γ, Λ be discrete groups that belong respectively to $\mathcal{C}_m$ and $\mathcal{C}_n$. Suppose that Γ and Λ sit as lattices in the same locally* *compact group. Then* $m = n$ *and* ρ_Γ *and* ρ_Λ *have commensurable images up to conjugation.*

Just as that two finitely generated groups that share a common cocompact envelope are QI, two groups that sit as lattices in the same locally compact group are measure-equivalent (ME) ([**Gro93**, 0.5.E]). It is worth mentioning that, contrary to the QI relation, the behaviour of the discrete groups in $T(n)$ with respect to the ME relation is not known (see [**Kid14, HR15**]). Recently new results in this setting have been announced by Gaboriau–Poulin–Tucker-Drob–Tserunyan–Wrobel.

APPENDIX A

Notions from topological dynamics

We refer to [**Gla76**] for a more comprehensive introduction to the notions presented here.

1. Preliminaries

Let G be a topological group. A non-empty space X is a G-space if X is endowed with a continuous action $G \times X \to X$. The action (or the G-space X) is **minimal** if all orbits are dense.

1.1. Stabilizers and neighbourhood stabilizer. Let X be a G-space. For $x \in X$, the stabilizer of x in G is the set of elements $g \in G$ such that $g(x) = x$. It is denoted G_x. The neighbourhood stabilizer of x in G is the set of elements $g \in G$ such that there exists a neighbourhood of x in X on which g acts trivially. It is denoted G_x^0.

1.2. Maps between compact spaces. A continuous surjective map $\pi : Y \to X$ between compact spaces is called **irreducible** if every proper closed subset of Y has a proper image in X. Equivalently, for every non-empty open subset U of Y, there exists $x \in X$ such that $\pi^{-1}(x) \subset U$. If X is a topological space, we denote by $R(X)$ the Boolean algebra of regular open subsets of X, and by $\hat{X}$ the Stone space of $R(X)$. When X is compact, there is an irreducible map $\hat{X} \to X$ (which is moreover universal with respect to irreducible maps $Y \to X$) [**Gle58**].

1.3. Extensions, factors. If X, Y are compact G-spaces and $\pi : Y \to X$ is a continuous surjective G-equivariant map, we say that X is a factor of Y, and that Y is an extension of X. If in addition $\pi : Y \to X$ is irreducible, then X is minimal if and only if Y is minimal. For X, Y minimal, $\pi : Y \to X$ is irreducible if and only if it is **highly proximal**, meaning that for every $x \in X$ the fiber $\pi^{-1}(x)$ is compressible [**AG77**]. A subset C of X is called **compressible** if there is $y \in Y$ such that for every neighbourhood U of y, there is $g \in G$ such that $g(C) \subset U$.

1.4. G-boundaries. If X is a compact space, we denote by $\mathrm{Prob}(X)$ the set of regular Borel probability measures on X, endowed with the weak*-topology ($\mathrm{Prob}(X)$ is seen as a subspace of the dual of the space of continuous functions on X). The space $\mathrm{Prob}(X)$ is a convex compact space, and the map $X \to \mathrm{Prob}(X)$ that associates to $x \in X$ the Dirac measure at x is continuous.

Let X be a compact G-space. The G-action on X is **proximal** if every pair of points in X is compressible. This is equivalent to asking that every finite subset of X is compressible. The G-action on X is **strongly proximal** if the closure of any G-orbit in $\mathrm{Prob}(X)$ contains a Dirac measure. Strongly proximal implies proximal. Strong proximality is stable under taking factors. The G-action on X is **extremely**

proximal if every proper closed subset of X is compressible. Away from the degenerate situation where X has cardinality two, extremely proximal implies strongly proximal.

We say that X is a G**-boundary** if X is both minimal and strongly proximal. There exists a unique G-boundary $\partial_F G$ with the property that any G-boundary is a factor of $\partial_F G$ [**Fur73**, Prop. 4.6]. This universal G-space $\partial_F G$ is referred to as the **Furstenberg boundary** of G. A locally compact group G is amenable if and only if every G-boundary is trivial (a one-point space), if and only if $\partial_F G$ is trivial [**Gla76**, III.3.1].

1.5. Profinite actions. Let X be a compact totally disconnected G-space. The G-action on X is profinite if the G-orbit of every clopen subset of X is finite. This is equivalent to asking that the G-action on X extends to a continuous action of the profinite completion of G. When X is metrizable, the G-action on X is profinite if and only if there exists a locally finite rooted tree on which G acts faithfully by automorphisms such that the G-space ∂T is isomorphic to X. See [**GNS00**, Proposition 6.4].

2. The Chabauty topology

If $\mathcal{X}$ is a locally compact space, we denote by $\mathcal{F}(\mathcal{X})$ the set of all closed subsets of $\mathcal{X}$. The sets

$$O(K; U_1, \dots, U_n) = \{C \in \mathcal{F}(\mathcal{X}) \,:\, C \cap K = \emptyset;\ C \cap U_i \neq \emptyset \text{ for all } i\},$$

where $K \subset \mathcal{X}$ is compact and $U_1, \dots, U_n \subset \mathcal{X}$ are open, form a basis for the Chabauty topology on $\mathcal{F}(\mathcal{X})$. Endowed with this topology, the space $\mathcal{F}(\mathcal{X})$ is compact. When $\mathcal{X}$ is discrete, $\mathcal{F}(\mathcal{X})$ is the set $\{0,1\}^{\mathcal{X}}$ of all subsets of $\mathcal{X}$, and the above topology is the product topology on $\{0,1\}^{\mathcal{X}}$.

The case of prime importance in this text is the one where $\mathcal{X} = G$ is a locally compact group. In that situation the space $\mathrm{Sub}(G)$ of closed subgroups of G is closed in $\mathcal{F}(G)$. In particular $\mathrm{Sub}(G)$ is a compact space.

3. Semi-continuous maps

Let X be a compact space and $\mathcal{X}$ a locally compact space. A map $\varphi\colon X \to \mathcal{F}(\mathcal{X})$ is **upper semi-continuous** if for every compact subset K of $\mathcal{X}$, $\{x \in X : \varphi(x) \cap K = \emptyset\}$ is open in X. It is **lower semi-continuous** if for every open subset U of $\mathcal{X}$, $\{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is open in X. Let $X_\varphi \subseteq X$ be the set of points where φ is continuous. When φ is either upper or lower semi-continuous, and $\mathcal{X}$ is second-countable, X_φ is a comeager subset of X [**Kur28**, Theorem VII].

We denote by $\eta : X \times \mathcal{F}(\mathcal{X}) \to X$ and $p : X \times \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{X})$ the projections to the first and second coordinate. For the following, see [**Gla75**, Theorem 2.3] and [**AG77**, Lemma I.1].

Proposition A.1. *Suppose X is a minimal compact G-space, $\mathcal{X}$ is a locally compact G-space, and $\varphi : X \to \mathcal{F}(\mathcal{X})$ is G-equivariant and upper or lower semi-continuous. Then the following hold:*

(i) The space

$$F_\varphi := \overline{\{(x, \varphi(x)) \,:\, x \in X\}} \subseteq X \times \mathcal{F}(\mathcal{X})$$

has a unique non-empty minimal closed G-invariant subset E_φ, and the space

$$T_\varphi := \overline{\{\varphi(x)\,:\,x \in X\}}$$

has a unique minimal closed G-invariant subset S_φ, and $p(E_\varphi) = S_\varphi$.

(ii) The extension $\eta : E_\varphi \to X$ is highly proximal.

If moreover $\mathcal{X}$ is second-countable, then

$$E_\varphi = \overline{\{(x, \varphi(x))\,:\,x \in X_\varphi\}},$$

and

$$S_\varphi = \overline{\{\varphi(x)\,:\,x \in X_\varphi\}}.$$

APPENDIX B

Micro-supported groups

Let G be a group and X a G-space. The rigid stabilizer $\mathrm{Rist}_G(U)$ of a subset $U \subseteq X$ is the pointwise fixator in G of the complement of U in X: $\mathrm{Rist}_G(U) = \mathrm{Fix}_G(X \backslash U)$.

Definition B.1. The G-action on X is **micro-supported** if G acts faithfully on X and $\mathrm{Rist}_G(U)$ acts non-trivially on X for every non-empty open subset U of X.

Since we are assuming the action is faithful, $\mathrm{Rist}_G(U)$ acts non-trivially on X if and only if $\mathrm{Rist}_G(U)$ is non-trivial. It is sometimes more convenient not to include faithfulness of the action in the definition, but for simplicity in this text we adopt the above definition. Being micro-supported is a very strong form of non-freeness of the action of G on X.

A group G is called micro-supported if G admits a micro-supported action. In that situation the group G does not necessarily admit a unique micro-supported action. In other words, the space X as in the definition and the G-action on X are not necessarily canonically attached to G. However, a reconstruction theorem due to Rubin asserts that G remembers (the Stone space of) the Boolean algebra of regular open subsets of X: if G admits micro-supported actions on X and Y, then there exists an isomorphism of G-spaces $\hat{X} \to \hat{Y}$ [**Rub96**]. When X, Y are compact minimal G-spaces, this theorem can equivalently be stated saying that X, Y admit a common highly proximal extension. Beyond this abstract result, in certain specific situations it happens that the group G actually admits a unique micro-supported action satisfying certain additional properties. Examples of results of this kind appear in [**Nek22**, Theorem 2.2.15], and [**LN02, GPS99a, Med11, Mat15**]. Reconstruction results of the same flavour are also known in the setting of groups of homeomorphisms and diffeomorphisms of manifolds [**Whi63, Fil82**]; or in the setting of group actions on measured spaces [**Dye59**].

There is an elementary observation in the study of normal subgroups of groups admitting a "sufficiently non-free" permutation action, which goes back at least to Higman (see the historical discussion in §2.2 in [**Nek22**]), and which is the common denominator of many proofs of simplicity. It is sometimes referred to as the "double commutator lemma", as it consists of a suitable manipulation involving iterated commutators of length two. In the setting of micro-supported groups, this lemma has been used extensively in order to show that many micro-supported groups are simple, or at least do not have many normal subgroups. In that setting it takes the following form ([**Nek22**, Proposition 2.2.4]):

Lemma B.2. *Suppose that G admits a micro-supported action on a Hausdorff space X, and N is a non-trivial normal subgroup of G. Then there is a non-empty open subset U of X such that N contains* $\mathrm{Rist}_G(U)'$.

Micro-supported groups are the main focus in Chapter 1 and Chapter 4. Examples appearing there include (this list is by no mean an exhaustive list of micro-supported groups):

Thompson's groups. The group F is the group of orientation preserving homeomorphisms g of the unit interval $[0,1]$ that are piecewise linear, with finitely many discontinuity points for the derivative, each one being a dyadic rational, and such that in restriction to each piece g has the form $x \mapsto 2^n x + q$ with $n \in \mathbb{Z}$ and q a dyadic rational. The groups T and V admit similar descriptions as group of homeomorphisms respectively of the circle and of the dyadic Cantor space. See [**CFP96**]. The group V admits higher dimensional generalizations nV, $n \geq 2$, acting on the Cantor n-cube [**Bri04**]. This family of groups admits a multitude of variations and generalizations, which we do not intend to list here.

Weakly branch groups. These are the groups that admit a profinite action on a compact metrizable totally disconnected space that is minimal and micro-supported. Equivalently, G is a weakly branch group if there exists a locally finite rooted tree on which G acts faithfully by automorphisms such that the G-action on the boundary ∂T is minimal and micro-supported. See [**BGS03**].

Topological full groups. Let Λ be a group acting minimally on the Cantor space X. The topological full group of this action, denoted $\mathbb{F}(\Lambda, X)$, is the group of homeomorphisms g of X such that for every $x \in X$ there exist a clopen neighbourhood U of x and an element $\gamma \in \Lambda$ such that $g(y) = \gamma(y)$ for every $y \in U$. Early studies of topological full groups were carried out in [**GPS99b**] and [**Mat06**]. See the survey [**Cor14**] for more recent developments, notably [**JM13**], and additional references.

Groups of piecewise projective homeomorphisms. Consider the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. Following [**Mon13**], if A is a subring of $\mathbb{R}$, we denote by $G(A)$ the group of homeomorphisms of $\mathbb{P}^1(\mathbb{R})$ which are piecewise $\mathrm{PSL}(2, A)$, each piece being a closed interval, with breakpoints in the set of ends of hyperbolic elements of $\mathrm{PSL}(2, A)$. Let also $H(A)$ be the stabilizer of the point ∞ in $G(A)$. The actions of $G(A)$ and $H(A)$ respectively on $\mathbb{P}^1(\mathbb{R})$ and $\mathbb{R}$ are micro-supported. Monod showed that when A is a dense subring of $\mathbb{R}$, the group $H(A)$ is non-amenable and yet does not contain any non-abelian free subgroups [**Mon13**].

Groups acting on trees with almost prescribed local action. Let $d \geq 3$, and let T_d be a d-regular tree. Let $F \lneq F' \leq \mathrm{Sym}(d)$ be two permutation groups such that F acts freely transitively. The group $G(F, F')$ is the group of automorphisms of T_d whose local actions are in F' for all vertices, and in F for all but finitely many vertices. See [**LB16**, Section 3] for a formal definition. The action of $G(F, F')$ on the boundary ∂T is micro-supported. These groups were the topic of Chapter 4 of the author's PhD thesis [**LB15**].

Groups of piecewise prescribed automorphisms of trees. Let T be a tree. To any subgroup $G \leq \mathrm{Aut}(T)$, one can associate the group $\mathrm{P}(G)$ of automorphisms of T acting piecewise like G [**LB17**]. The formal definition is as follows. If A is a finite subtree of T, we let δA be the vertices of A having at least one neighbour that is not in A. For $v \in \delta A$, denote by T_v the subtree of T made of vertices whose projection on A is the vertex v. The subgroup $\mathrm{P}(G)$ of $\mathrm{Aut}(T)$ consists of elements $\gamma \in \mathrm{Aut}(T)$ such that there exists a finite subtree A of T such that for every $v \in \delta A$ there exists $g_v \in G$ such that γ and g_v coincide on T_v. If the action on G on T is minimal and of general type and stabilizers of vertices in G are non-trivial, then the action of $\mathrm{P}(G)$ on ∂T is micro-supported [**LB17**, Section 4].

Neretin groups of almost automorphisms of trees. Let $d, k \geq 2$ be integers, and let $\mathcal{T}_{d,k}$ be a rooted tree such that the root has k descendants, and every vertex distinct from the root has d descendants. We consider the triples of the form (A, B, φ), where A and B are finite subtrees of $\mathcal{T}_{d,k}$ containing the root such that $|\partial A| = |\partial B|$, and φ is a forest isomorphism between $\mathcal{T}_{d,k} \setminus A$ and $\mathcal{T}_{d,k} \setminus B$. Two triples (A, B, φ) and (A', B', φ') are equivalent if there exists some finite subtree A'' containing $A \cup A'$ and such that φ and φ' coincide on $\mathcal{T}_{d,k} \setminus A''$. The group of almost automorphisms of $\mathcal{T}_{d,k}$, denoted by $\mathcal{N}_{d,k}$, is the quotient of the set of all (A, B, φ) by the above equivalence relation. It is naturally a group of homeomorphisms of the boundary $\partial \mathcal{T}_{d,k}$. The groups $\mathcal{N}_{d,k}$ are compactly generated, simple, totally disconnected locally compact groups [**Kap99, CDM11**]. These groups were the topic of Chapter 3 of the author's PhD thesis [**LB15**]. See [**BCGM12, ST17, Zhe19b**] for remarkable properties of this family of groups.

APPENDIX C

Geometric actions on metric spaces, commability

We refer to [**CH16, DK18**] for details and a more comprehensive introduction to the notions mentioned here.

Let X be a proper metric space. A continuous action of a locally compact group on a proper metric space X is **geometric** if it is isometric, proper and cocompact. Examples include:

- if Γ is a finitely generated discrete group, the action of Γ on any Cayley graph $\mathrm{Cay}(\Gamma, S)$ of Γ associated to a finite generating subset S, is geometric.
- If G is a compactly generated totally disconnected locally compact group, the action of G on any Cayley-Abels graph $\mathrm{CayAb}(G, U, S)$ of G associated to a compact open subgroup U and a compact generating subset S (which is required to be U-bi-invariant), is geometric. Recall that the vertex set of $\mathrm{CayAb}(G, U, S)$ is the coset space G/U, and two cosets g_1U and g_2U are connected by an edge if there is $s \in S^{\pm 1}$ such that $g_2 = g_1 s$. The graph $\mathrm{CayAb}(G, U, S)$ is connected and locally finite. When G is discrete and U is the trivial subgroup, this reduces to the previous item. Cayley-Abels graphs exist more generally for locally compact groups G such that the identity component G^0 is compact.
- If G is a connected Lie group, K is a compact subgroup of G, and d a G-invariant Riemannian metric on G/K, the G-action on $(G/K, d)$ is geometric.

If X is a proper metric space, the group $\mathrm{Isom}(X)$ of bijective isometries of X, equipped with the compact-open topology, is a second countable locally compact group. If a locally compact group G admits a geometric action on X, then the associated homomorphism $G \to \mathrm{Isom}(X)$ is continuous, has a compact kernel, and a closed cocompact image. Hence $G \to \mathrm{Isom}(X)$ can be seen as the composition of the factor homomorphism $G \to G/K$ where K is the compact kernel, and the injective homomorphism from G/K in $\mathrm{Isom}(X)$, which has a closed cocompact image. The existence of a geometric action of G on X implies that G is σ-compact.

Definition C.1. The relation of **commability** among locally compact groups is the equivalence relation generated by $G \sim G/K$ for K a compact normal subgroup of G, and $G \sim H$ for H a closed cocompact subgroup of G.

For σ-compact groups (which includes the case of compactly generated groups), commability can be characterized as follows: G, H are commable if and only if there exist locally compact groups $G_0, \dots, G_n$ with $G_0 = G$ and $G_n = H$ such that for each i there exists a proper metric space on which G_i and G_{i+1} both act geometrically. In other words, commability is the transitive closure of the relation of having a geometric action on the same proper metric space. We note that the relation of having a geometric action on the same proper metric space is clearly reflexive and symmetric, but is indeed not transitive, as follows from the existence of two finitely generated virtually free groups with no geometric action on the same space [**MSW03**, Corollary 10].

Two groups are called virtually isomorphic if they have finite index subgroups that are isomorphic, and virtually isomorphic up to finite kernel if they have finite index subgroups that are isomorphic after modding out by a finite normal subgroup. Groups that are virtually isomorphic up to finite kernel are obviously commable.

If G is a compactly generated locally compact group, G can be equipped with the word metric associated to some compact generating subset. Two compact generating subsets yield bi-Lipschitz equivalent word metrics. In particular G has a well-defined quasi-isometry type. If two compactly generated locally compact groups G, H act geometrically on the same proper metric space, then the locally compact version of the Milnor-Schwarz lemma asserts that G, H are quasi-isometric (QI). This fundamental observation is the source of most familiar instances of quasi-isometries between groups. Since being QI is an equivalence relation, two commable compactly generated groups are QI. So the commability relation sits between the relation of being virtually isomorphic up to finite kernel and the QI relation (but is in spirit closer to the QI relation).

For a finitely generated discrete group Γ, the QI-rigidity problem for Γ asks for a description of all groups that are QI to Γ. That problem is sometimes restricted to the realm of discrete groups (i.e. describe all finitely generated discrete groups QI to Γ). See [**DK18**, Chapter 25] for a survey on QI-rigidity. Similarly, the commability problem for Γ asks for a description of all finitely generated discrete groups, or more generally compactly generated locally compact groups, commable to Γ. That problem admits the following sub-problems:

(1) *Describe all locally compact groups G that contain a discrete and cocompact subgroup isomorphic to* Γ. Such a group G is called a cocompact envelope of Γ. If Γ acts faithfully and geometrically on a proper metric space X, then the group $\mathrm{Isom}(X)$ is a cocompact envelope of Γ. Hence, although we are more focussed on the groups rather than the spaces, (1) is tightly connected to the study of proper metric spaces on which Γ can act faithfully and geometrically.

(2) *Describe the discrete groups Λ such that there is a locally compact group G such that G is a common cocompact envelope of* Γ *and* Λ. Equivalently, these are the groups Λ such that there is a proper metric space on which Γ and Λ both act faithfully and geometrically. We note that in such a situation the groups Γ and Λ are then measure-equivalent (ME), so (2) is not only connected to QI-rigidity, but also to ME-rigidity.

The above problems are also often considered when Γ is not a fixed group, but a member of a class of groups $\mathcal{C}$. In that realm the QI-rigidity problem (resp. commability problem) asks whether the class $\mathcal{C}$ is stable under QI (resp. under commability). Similarly (1) and (2) above can be considered for a class $\mathcal{C}$ rather than a single group. The results presented in Chapter 5 and Chapter 6 fall into this setting.

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