THE COMMENSURATOR OF A COCOMPACT LATTICE IN THE AUTOMORPHISM GROUP OF A REGULAR TREE IS NOT VIRTUALLY SIMPLE

Abstract. We show that the commensurator of a cocompact lattice in the automorphism group $\mathrm{Aut}(T_k)$ of a regular tree T_k is not virtually simple.

Let $k \geq 3$, and T_k the k-regular tree. We denote by $\mathrm{Aut}(T_k)$ the group of automorphisms of the tree T_k. With its natural topology, $\mathrm{Aut}(T_k)$ is a totally disconnected locally compact group.

Theorem 1. *Let Γ be a cocompact lattice in $G = \mathrm{Aut}(T_k)$. Then the commensurator $\mathrm{Comm}_G(\Gamma)$ of Γ in G is not virtually simple. More precisely, $\mathrm{Comm}_G(\Gamma)$ admits a proper quotient with infinite order elements.*

For Γ, G as in the theorem, the subgroup $\mathrm{Comm}_G(\Gamma)$ is a dense subgroup of G [BK90, LMZ94]. The question whether the type preserving index two subgroup of $\mathrm{Comm}_G(\Gamma)$ is simple had been raised by Lubotzky–Mozes–Zimmer [LMZ94]. See Section 2 for previous results regarding this question.

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1. Preliminaries on abstract commensurators

Let A be a finite index subnormal subgroup of a group Γ. We denote by $CF(\Gamma; A)$ the set of isomorphism classes of composition factors appearing in some (equivalently, any) composition series which starts from A and ends with Γ [BELB$^+$25]. Each composition factor appears with a certain multiplicity; and $CF(\Gamma; A)$ takes into account these multiplicities.

Lemma 2. *Let A, B, C, D be finite index subnormal subgroups of Γ, and suppose $\varphi : A \to B$ and $\psi : C \to D$ are isomorphisms. Suppose $CF(\Gamma; A) = CF(\Gamma; B)$, and suppose $[\varphi] = [\psi]$ in the commensurator $\mathrm{Comm}(\Gamma)$ of Γ. Then $CF(\Gamma; C) = CF(\Gamma; D)$.*

Proof. Take E a finite index subgroup of $A \cap C$ on which φ and ψ coincide (actually we will only use that $\varphi(E)$ is equal to $\psi(E)$). Upon passing to a smaller subgroup one can assume E is normal in Γ. Set $F := \varphi(E) = \psi(E)$. Since E is normal in A, F is normal in B. So F is subnormal in Γ. One has $CF(\Gamma; E) = CF(\Gamma; A) \sqcup CF(A; E)$ and $CF(\Gamma; F) = CF(\Gamma; B) \sqcup CF(B; F)$. Using $CF(\Gamma; A) = CF(\Gamma; B)$ and $CF(A; E) = CF(B; F)$ (since φ is an isomorphism between A and B that sends E to F), we infer $CF(\Gamma; E) = CF(\Gamma; F)$. This can be rewritten as $CF(\Gamma; C) \sqcup CF(C; E) = CF(\Gamma; D) \sqcup CF(D; F)$. Now ψ is an isomorphism between C and D that sends E to F, so $CF(C; E) = CF(D; F)$. Combined with the previous equality, this yields

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$CF(\Gamma;C)\sqcup CF(C;E)=CF(\Gamma;D)\sqcup CF(C;E)$. Removing $CF(C;E)$ in this equality is allowed, and yields $CF(\Gamma;C)=CF(\Gamma;D)$. □

We recall two definitions from [BELB$^+$25]:

- For each finite index subgroup H of Γ, there is a natural homomorphism $\mathrm{Aut}(H)\to\mathrm{Comm}(\Gamma)$, and $\mathrm{AComm}(\Gamma)$ is the subgroup of $\mathrm{Comm}(\Gamma)$ generated by the images of $\mathrm{Aut}(H)\to\mathrm{Comm}(\Gamma)$ when H ranges over finite index subgroups of Γ.
- $\mathrm{Comm}_{SN}(\Gamma)$ is the set of elements of $\mathrm{Comm}(\Gamma)$ which can be represented by an isomorphism $\varphi : A\to B$ such that A and B are finite index subnormal subgroups of Γ such that $CF(\Gamma;A)=CF(\Gamma;B)$.

[BELB$^+$25, Lemma 2.22] ensures $\mathrm{AComm}(\Gamma)$ is a normal subgroup of $\mathrm{Comm}(\Gamma)$ provided Γ is finitely generated. And [BELB$^+$25, Theorem 5.5] shows that

(1) $\mathrm{Comm}_{SN}(\Gamma)$ is a subgroup of $\mathrm{Comm}(\Gamma)$.
(2) $\mathrm{AComm}(\Gamma)$ is contained in $\mathrm{Comm}_{SN}(\Gamma)$ provided Γ is finitely generated.

If A,B are finite index subnormal subgroups of a group Γ, we write $CF(\Gamma;A)\preccurlyeq CF(\Gamma;B)$ if every composition factor appearing in $CF(\Gamma;A)$ also appears in $CF(\Gamma;B)$. This definition only asks that composition factors appear; it does not take multiplicities into account. We say that $CF(\Gamma;A)$ and $CF(\Gamma;B)$ are equivalent if $CF(\Gamma;A)\preccurlyeq CF(\Gamma;B)$ and $CF(\Gamma;B)\preccurlyeq CF(\Gamma;A)$.

Proposition 3. *Let Γ be a finitely generated group. Let A,B be finite index subnormal subgroups of Γ, and suppose $\varphi : A\to B$ is an isomorphism. If $CF(\Gamma;A)$ and $CF(\Gamma;B)$ are not equivalent, then the element $[\varphi]$ of $\mathrm{Comm}(\Gamma)$ has infinite order in the quotient $\mathrm{Comm}(\Gamma)/\mathrm{AComm}(\Gamma)$.*

Proof. We define inductively a sequence (A_n,B_n,f_{n+1}) where A_n,B_n are finite index subnormal subgroups of Γ, $f_{n+1}:A_n\to B_n$ is an isomorphism such that $[f_{n+1}]=[\varphi]^{n+1}$ in $\mathrm{Comm}(\Gamma)$, $CF(\Gamma;A_n)$ and $CF(\Gamma;A)$ are equivalent and $CF(\Gamma;B_n)$ and $CF(\Gamma;B)$ are equivalent. We set $A_0=A$, $B_0=B$ and $f_1=\varphi$. Now suppose (A_n,B_n,f_{n+1}) have been constructed. We verify that $A_{n+1}=\varphi^{-1}(A_n\cap B)$, $B_{n+1}=f_{n+1}(A_n\cap B)$, and $f_{n+2}=f_{n+1}\circ\varphi : A_{n+1}\to B_{n+1}$ verify the required properties. Note that B_{n+1} is a subgroup of B_n. By definition of the composition in $\mathrm{Comm}(\Gamma)$ one has $[f_{n+2}]=[f_{n+1}][\varphi]$. Hence $[f_{n+2}]=[\varphi]^{n+1}[\varphi]=[\varphi]^{n+2}$. Since A_n is subnormal in Γ, $B\cap A_n$ is subnormal in B, and consequently $A_{n+1}=\varphi^{-1}(A_n\cap B)$ is subnormal in $\varphi^{-1}(B)=A$. Hence A_{n+1} is subnormal in Γ. Similarly B_{n+1} is subnormal in B_n, and hence in Γ. Since $A_{n+1}\leq A\leq\Gamma$, we have $CF(\Gamma;A_{n+1})=CF(\Gamma;A)\sqcup CF(A;A_{n+1})$. Since φ is an isomorphism from A to B sending A_{n+1} to $B\cap A_n$, we also have $CF(A;A_{n+1})=CF(B;B\cap A_n)$. Now intersecting with B a composition series associated to A_n, it is easily seen that $CF(B;B\cap A_n)\preccurlyeq CF(\Gamma;A_n)$. Since $CF(\Gamma;A_n)\preccurlyeq CF(\Gamma;A)$, we deduce $CF(B;B\cap A_n)\preccurlyeq CF(\Gamma;A)$, and hence $CF(\Gamma;A_{n+1})\preccurlyeq CF(\Gamma;A)$. Since we also have $CF(\Gamma;A)\preccurlyeq CF(\Gamma;A_{n+1})$, it follows that $CF(\Gamma;A)$ and $CF(\Gamma;A_{n+1})$ are equivalent. A similar argument yields $CF(\Gamma;B)$ and $CF(\Gamma;B_{n+1})$ are equivalent. We have thus verified all the properties.

Now assume $CF(\Gamma;A)$ and $CF(\Gamma;B)$ are not equivalent. Then $CF(\Gamma;A_n)$ and $CF(\Gamma;B_n)$ are not equivalent, and hence are not equal. Lemma 2 therefore implies $[f_{n+1}]$ does not belong to $\mathrm{Comm}_{SN}(\Gamma)$. Since $\mathrm{AComm}(\Gamma)$ is contained in $\mathrm{Comm}_{SN}(\Gamma)$, the image of $[f_{n+1}]$ is non-trivial in the quotient $\mathrm{Comm}(\Gamma)/\mathrm{AComm}(\Gamma)$.

Since $[f_{n+1}] = [\varphi]^{n+1}$, this shows the image of $[\varphi]$ has infinite order in $\mathrm{Comm}(\Gamma)/\mathrm{AComm}(\Gamma)$. □

2. The commensurator of a cocompact lattice in the automorphism group of a regular tree

Let $k \geq 3$, and T_k the k-regular tree. Let W be the free product of k copies of the cyclic group of order 2 (the free Coxeter group of rand k). We fix $a_1, \ldots, a_k \in W$ such that $a_i^2 = 1$ and $a_1, \ldots, a_k$ generate W. We identify the tree T_k with the Cayley graph of W associated to $a_1, \ldots, a_k$. This means that there is a vertex v_0 in T_k corresponding to the identity element, and that there is a coloring of the edges of T_k by the elements of $\{1, \ldots, k\}$, where an edge is colored i if it corresponds to an edge between two elements γ and γa_i of W. In particular we view W as a subgroup of $\mathrm{Aut}(T_k)$. It is a cocompact lattice because W acts freely and transitively on vertices of T_k. Note that W is exactly the subgroup of $\mathrm{Aut}(T_k)$ preserving the coloring.

Following [LMZ94], we denote by C_k the commensurator of W in $\mathrm{Aut}(T_k)$. Any cocompact lattice Γ of $\mathrm{Aut}(T_k)$ admits a conjugate that is commensurable with W in $\mathrm{Aut}(T_k)$, and hence has a commensurator in $\mathrm{Aut}(T_k)$ that is conjugate to C_k [BK90, 4.15-4.17]. Hence Theorem 1 is equivalent to the corresponding assertion about the group C_k. The group C_k is known to be monolithic (i.e. admits a non-trivial normal subgroup contained in any other non-trivial normal subgroup) [LMZ94, Cap20]. Moreover Lubotzky–Mozes–Zimmer showed that the monolith M_k of C_k contains the type preserving index two subgroup of W [LMZ94, Proposition 5.1], and Caprace showed that M_k is a simple group [Cap20, Theorem A.1]. Theorem 1 asserts that M_k has infinite index in C_k, and the quotient C_k/M_k admits infinite order elements.

We will make use of the interpretation of elements of the stabilizer of v_0 in C_k as recoloring of finite graphs. The setting is the following. Let X be a finite connected k-regular graph (every vertex has k adjacent edges). Edges are non-oriented. We allow an edge to be a loop, and we also allow multiple edges. By a coloring c of X we mean an assignment for each edge of X of an element in $\{1, \ldots, k\}$ (the color of that edge) such that for every vertex, the k adjacent edges have different colors. If c is a coloring of X and x_0 a vertex of X, there exists a unique color preserving map $\pi : T_k \to (X, c)$ such that $\pi(v_0) = x_0$, and this map is surjective.

Associated to a coloring of X there is an action of W on the vertex set VX. It is defined by declaring that a_i flips all pairs of distinct vertices joined by an edge with color i, and a_i fixes all vertices for which the adjacent edge with color i is a loop. By the universal property of free products, this indeed defines an action of W on VX. Note that the graph and the coloring encode the action on VX, but the action needs not preserve the graph structure.

We refer to Section 2 (notably to Propositions 2.2 and 2.4) in [LMZ94] for the following.

Proposition 4. *Let X be a finite connected k-regular graph, and x_0 a vertex of X. Let c_1, c_2 be two colorings of X, and for $i = 1, 2$ let H_i be the stabilizer of x_0 in W for the action of W on VX associated to c_i. Then one can associate to the pair (c_1, c_2) an element $g \in \mathrm{Aut}(T_k)$ such that $g(v_0) = v_0$ and $gH_1g^{-1} = H_2$ (in particular g belongs to C_k).*

The definition of g goes as follows. The pair (c_1, c_2) gives rise to a collection of permutations $(\sigma_x)_{x \in VX}$ of $\{1, \ldots, k\}$, defined by the property that for every $x \in VX$ and every edge e adjacent to x, the coloring c_2 is given by $c_2(e) = \sigma_x(c_1(e))$. The element g associated to X, x_0, c_1, c_2 is then the element of $\mathrm{Aut}(T_k)$ determined by the condition $g(v_0) = v_0$ and the local action of g around a vertex v of T_k is given by the permutation $\sigma_{\pi(v)}$, where $\pi : T_k \to (X, c_1)$ is the unique color preserving map such that $\pi(v_0) = x_0$. We refer to [LMZ94] for details.

Proof of Theorem 1. As explained above, one has to show C_k has a proper quotient with infinite order elements. Since the only element of $\mathrm{Aut}(T_k)$ centralizing a finite index subgroup of W is the identity, the natural homomorphism $C_k \to \mathrm{Comm}(W)$ is injective [BK90, B.7]. In the sequel we identify C_k with its image in $\mathrm{Comm}(W)$. We consider the image of C_k in $\mathrm{Comm}(W)/\mathrm{AComm}(W)$. Since C_k intersects $\mathrm{AComm}(W)$ along an infinite normal subgroup (because W is contained in $\mathrm{AComm}(W)$), to obtain the conclusion it is enough to show that the image of C_k in $\mathrm{Comm}(W)/\mathrm{AComm}(W)$ admits infinite order elements.

We fix an integer $s \geq 3$, and let $r = 2^s$. Let X be the k-regular graph with r vertices $x_0, \ldots, x_{r-1}$, with an edge e_i between x_i and x_{i+1} for every $i = 0, \ldots, r-2$, and such that all other edges are loops. We consider a coloring c_1 of X such that edges $e_0, e_2, \ldots, e_{r-2}$ have color 1, and edges $e_1, e_3, \ldots, e_{r-3}$ have color 2 (how c_1 is defined on other edges will not matter). The permutation group on VX induced by the W-action associated to c_1 is the dihedral group of order $2r$. In particular it is a 2-group. Hence the stabilizer H_1 of x_0 in W is a subnormal subgroup of W, and $CF(W; H_1)$ is made of the cyclic group of order 2 with multiplicity $r/2$.

We now define a coloring c_2 of X by starting from c_1, and specifying for each vertex x of X a permutation of $\{1, \ldots, k\}$. For vertices $x_3, \ldots, x_{r-1}$, we take the identity. In other words c_2 coincide with c_1 on all edges adjacent to those vertices. For x_0, x_1, x_2, we take respectively $(12), (123), (23)$. So exactly five edges have changed color, among which e_0, e_1, which were previously colored 1 and 2 and are now colored 2 and 3. The element a_3 acts on VX as a transposition, and the supports of a_1 and a_3 intersect along a singleton, and similarly for a_2 and a_3. One then easily verifies that the permutation group on VX induced by the W-action associated to c_2 is the symmetric group $\mathrm{Sym}(r)$. As before we denote by H_2 the stabilizer of x_0 in W.

Proposition 4 applied to X, x_0, c_1, c_2 provides $g \in C_k$ such that $gH_1g^{-1} = H_2$. Let $B = \mathrm{Core}_W(H_2)$, and $A = g^{-1}Bg$, so that conjugation by g defines an isomorphism $A \to B$. The subgroup B is normal in W, and $CF(W; B)$ contains the cyclic group of order 2 and $\mathrm{Alt}(r)$ (because $s \geq 3$), both with multiplicity 1. The subgroup A is normal in H_1, and H_1 is subnormal in W, so A is subnormal in W. And by the description of $CF(W; H_1)$ given above, we infer that $CF(W; A)$ contains the cyclic group of order 2 with multiplicity at least $r/2$. Cardinality of the index of A in W (which is $r!$) therefore prevents $\mathrm{Alt}(r)$ from appearing in $CF(W; A)$. Hence $CF(W; A)$ and $CF(W; B)$ are not equivalent. Proposition 3 applies, and asserts that the image of g in $\mathrm{Comm}(W)/\mathrm{AComm}(W)$ has infinite order. □

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